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An inexact parallel splitting augmented Lagrangian method for monotone variational inequalities with separable structures

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Abstract Splitting methods have been extensively studied in the context of convex programming and variational inequalities with separable structures. Recently, a parallel splitting method based on the augmented Lagrangian method (abbreviated as PSALM) was proposed in He (Comput. Optim. Appl. 42:195–212, 2009) for solving variational inequalities with separable structures. In this paper, we propose the inexact version of the PSALM approach, which solves the resulting subproblems of PSALM approximately by an inexact proximal point method. For the inexact PSALM, the resulting proximal subproblems have closed-form solutions when the proximal parameters and inexact terms are chosen appropriately. We show the efficiency of the inexact PSALM numerically by some preliminary numerical experiments.

Keywords Variational inequalities · Splitting method · Parallel method · Proximal point method · Augmented Lagrangian method · Prediction-correction method

1 Introduction

Let $\Omega \subset \mathcal{R}^n$ be a nonempty closed convex set and F be a continuous mapping from $\mathcal{R}^n$ into itself. The variational inequality (VI) problem, denoted by VI(Ω, F), is to find $u \in \Omega$ such that

$$(u' - u)^T F(u) \geq 0, \quad \forall\, u' \in \Omega, \tag{1.1}$$

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where "T" denotes the standard inner product. In this paper, we consider the VI(Ω, F) with the following separable structure:

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix}, \tag{1.2}$$

and

$$\Omega := \{(x, y) \mid Ax + By = b,\ x \in \mathcal{X},\ y \in \mathcal{Y}\}, \tag{1.3}$$

where $\mathcal{X} \subset \mathcal{R}^{n_1}$ and $\mathcal{Y} \subset \mathcal{R}^{n_2}$ are nonempty closed and convex sets; $A \in \mathcal{R}^{m \times n_1}$ and $B \in \mathcal{R}^{m \times n_2}$ are given matrices; $f : \mathcal{X} \to \mathcal{R}^{n_1}$ and $g : \mathcal{Y} \to \mathcal{R}^{n_2}$ are given monotone mappings; $b \in \mathcal{R}^m$ is a given vector and $n_1 + n_2 = n$. For wide applications of VI(Ω, F) with the separable structure (1.2)–(1.3), see e.g. [2, 11, 19].

The favorable separable structure of (1.2)–(1.3) has inspired many splitting type methods, with the main purpose of exploiting the properties of f and g individually and separably. Such a splitting method is the alternating direction method (ADM) which was proposed in [8] and studied intensively in the literature, see e.g. [4, 5, 7, 9–11, 13]. Recently, the so-called parallel splitting augmented Lagrangian method (PSALM for abbreviation) was proposed in [16] for solving the VI (1.1)–(1.3). With the given iterate $w^k = (x^k, y^k, \lambda^k)$, at each iteration PSALM requires to solve the following sub-VIs:

$$x \in \mathcal{X}, \quad (x' - x)^T \{ f(x) - A^T[\lambda^k - H(Ax + By^k - b)] \} \geq 0, \quad \forall\ x' \in \mathcal{X}, \tag{1.4a}$$

$$y \in \mathcal{Y}, \quad (y' - y)^T \{ g(y) - B^T[\lambda^k - H(Ax^k + By - b)] \} \geq 0, \quad \forall\ y' \in \mathcal{Y}, \tag{1.4b}$$

where λ^k is the Lagrange multiplier associated with the linear constraint in (1.3) and $H \in \mathcal{R}^{m \times m}$ is a positive definite matrix which plays the role of the penalty parameter for the violation of the linear constraint in (1.3). After (1.4), PSALM generates the new iterate $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ via updating the Lagrange multiplier with the output of (1.4) and then implementing an additional correction step. Therefore, PSALM is in the nature of a prediction-correction type method. Retaining the capability of exploiting properties of f and g individually and separably, PSALM differs significantly from other splitting methods like ADM in that the resulting subproblems (1.4a) and (1.4b) can be computed in parallel. This parallel advantage is particularly interesting when the dimensionality of data is tremendously large. We refer to, e.g. [1, 17] and reference cited therein, for some recent development on parallel splitting methods.

The convergence of PSALM was established in [16] under the assumption that the sub-VIs in (1.4) should be solved exactly. However, unless f and g have very particular structures, the sub-VIs in (1.4) could be too difficult to be solved exactly. Thus, iterative subroutines are required to obtain approximate solutions of these sub-VIs. This difficulty excludes the direct application of PSALM for some cases where the subproblems in (1.4) have no closed-form solutions, and it urges research on inexact versions of the PSALM approach which solve (1.4) approximately subject to certain inexact criterion. This is the main motivation of the paper. On the other hand, recall that the subproblems in (1.4) actually dominate the computation of PSALM at each iteration. Thus, we are interested in finding a way to solve (1.4) approximately and efficiently. In particular, we apply the classical proximal point algorithm (PPA) [18, 20] to regularize the resulting sub-VIs in (1.4), and then we solve the proximally regularized sub-VIs approximately. We will show that the regularized sub-VIs have closed-form solutions provided that the proximal parameters and inexact criteria are chosen appropriately. Through this strategy, the difficulty of solving (1.4) exactly can be alleviated substantially, and the resulting inexact PSALM can be easily implementable.

We concentrate our discussion on the case where the mapping $f(x)$ and $g(y)$ are "black-box" mappings in the sense that only mapping values of f and g can be evaluated, while derivative information of these mappings are unavailable. A good example to illustrate this scenario is that for some concrete applications of the VI (1.1)–(1.3) arising in economics and transportation [19], the mappings $f(x)$ and $g(y)$ are in the equilibrium nature which essentially implies that the decision-makers can only know mapping values of f and g via experimental observations, without knowing their mathematical expressions.

The rest of this paper is organized as follows. In Sect. 2, we review some preliminaries which are useful for further analysis. In Sect. 3, we first analyze the motivation of the inexact PSALM and elucidate the strategy of solving the resulting subproblems. Then, we present the algorithm of the inexact PSALM and give some remarks. Last, we discuss briefly the relationship between the inexact PSALM and some existing methods. In Sect. 4, we establish the convergence of the inexact PSALM. In Sect. 5, we delineate the implementation of the inexact PSALM and report some numerical results when it is applied to solve some traffic equilibrium problems. Finally, we make some conclusions in Sect. 6.

Throughout, we make the following assumptions:

A.1 The sets $\mathcal{X}$ and $\mathcal{Y}$ are simple in the sense that it is easy to compute the projection onto them under the Euclidean norm. Such simple sets include the positive orthant, balls or boxes.

A.2 The mappings $f(x)$ and $g(y)$ are Lipschitz continuous on $\mathcal{X}$ and $\mathcal{Y}$, respectively; while the Lipschitz constants are not necessarily known.

A.3 The solution set of the VI (1.1)–(1.3) is nonempty.

2 Preliminaries

In this section, we summarize some basic properties and related definitions which will be used in the following discussions.

The Euclidean norm of $v \in \mathcal{R}^n$ is defined by $\|v\| = \sqrt{v^T v}$. For the nonempty closed convex subset $\Omega \subset \mathcal{R}^n$, we denote by $P_\Omega(\cdot)$ the projection onto Ω under the Euclidean norm:

$$P_\Omega(v) = \arg\min\{\|v - u\| \,|\, u \in \Omega\}.$$

Then, some important inequalities regarding the projection operator are summarized in the following lemma, whose proof can be found in [2].

Lemma 2.1 *Let $\Omega \subset \mathcal{R}^n$ be nonempty, closed and convex. Let $P_\Omega(\cdot)$ be the projection operator onto Ω under the Euclidean norm. Then, we have*

$$(v - P_\Omega(v))^T (P_\Omega(v) - u) \geq 0, \quad \forall\, v \in R^n, \quad \forall\, u \in \Omega; \tag{2.1}$$

$$\|P_\Omega(v) - P_\Omega(w)\| \leq \|v - w\|, \quad \forall\, v,\ w \in R^n; \tag{2.2}$$

$$\|P_\Omega(v) - u\|^2 \leq \|v - u\|^2 - \|v - P_\Omega(v)\|^2, \quad \forall\, v \in R^n, \quad \forall\, u \in \Omega. \tag{2.3}$$

We recall the definitions of monotone and strongly monotone mappings.

Definition 2.1 A mapping $F : R^n \to R^n$ is said to be

(a) monotone on Ω if

$$(x - y)^T (F(x) - F(y)) \geq 0, \quad \forall\, x, y \in \Omega;$$

(b) strongly monotone with the modulus $\mu > 0$ on Ω if

$$(x - y)^T (F(x) - F(y)) \geq \mu \|x - y\|^2, \quad \forall\, x, y \in \Omega.$$

The following lemma states an important result which characterizes a VI by a projection equation, and its proof can be found in [2, pp. 267].

Lemma 2.2 *Let $\Omega \subset \mathcal{R}^n$ be nonempty, closed and convex. Let $P_\Omega(\cdot)$ be the projection operator onto Ω under the Euclidean norm. Then, u^* is a solution of VI(Ω, F) if and only if it satisfies*:

$$u^* = P_\Omega[u^* - \beta F(u^*)], \quad \forall\, \beta > 0. \tag{2.4}$$

As shown in [16], we can reformulate the VI (1.1)–(1.3) into a compact VI. More specifically, by attaching the Lagrange multiplier $\lambda \in \mathcal{R}^m$ to the linear constraints $Ax + By = b$, the VI (1.1)–(1.3) amounts to finding $(x, y, \lambda) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^m$ such that

$$\begin{cases} (x' - x)^T \{f(x) - A^T \lambda\} \geq 0, \\ (y' - y)^T \{g(y) - B^T \lambda\} \geq 0, \qquad \forall\, (x', y') \in \mathcal{X} \times \mathcal{Y}. \\ Ax + By - b = 0, \end{cases} \tag{2.5}$$

Equivalently, (2.5) is the following VI denoted by VI$(Q, \mathcal{W})$:

$$(w' - w)^T Q(u) \geq 0, \quad \forall\, w' \in \mathcal{W}, \tag{2.6}$$

where

$$\mathcal{W} := \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^m \quad \text{and} \quad Q(w) := \begin{pmatrix} f(x) - A^T \lambda \\ g(y) - B^T \lambda \\ Ax + By - b \end{pmatrix}. \tag{2.7}$$

Note that the mapping Q is monotone whenever f and g are monotone. In addition, under the assumption **A.3**, the solution set of VI$(Q, \mathcal{W})$, denoted by $\mathcal{W}^*$, is also nonempty. In fact, according to [6], $\mathcal{W}^*$ is closed and convex since the mappings $f(x)$ and $g(y)$ are monotone, and the sets $\mathcal{X}$ and $\mathcal{Y}$ are closed and convex.

Based on the fact that VI (1.1)–(1.3) is equivalent to VI$(Q, \mathcal{W})$, the following lemma is an immediate conclusion of Lemma 2.2.

Lemma 2.3 *Solving the VI* (1.1)–(1.3) *amounts to seeking a zero point of the mapping*

$$E_{[\mathcal{W},\, Q]}(w) := \begin{pmatrix} x - P_{\mathcal{X}}\{x - [f(x) - A^T\lambda]\} \\ y - P_{\mathcal{Y}}\{y - [g(y) - B^T\lambda]\} \\ Ax + By - b \end{pmatrix} = w - P_{\mathcal{W}}[w - Q(w)]. \tag{2.8}$$

3 Inexact parallel splitting augmented Lagrangian methods for VI (1.1)–(1.3)

In this section, we propose the inexact PSALM for VI(1.1)–(1.3), and analyze its relationship with some existing methods. First of all, we explain the motivation of the inexact PSALM.

3.1 Motivation

With the aforementioned preliminaries in Sect. 2, let us revisit the resulting subproblems in (1.4). Then, our idea of developing implementable inexact PSALM will be clear. In particular, by applying Lemma 2.2, we know that the solutions of the subproblems (1.4a) and (1.4b) are characterized respectively by

$$x = P_{\mathcal{X}}\big[x - \beta[f(x) - A^T[\lambda^k - H(Ax + By^k - b)]]\big] \tag{3.1}$$

and

$$y = P_{\mathcal{Y}}\big[y - \beta[g(y) - B^T[\lambda^k - H(Ax^k + By - b)]]\big], \tag{3.2}$$

with any $\beta > 0$. The equation (3.1) (Resp. (3.2)), however, is implicit in the sense that the variable x (Resp. y) appears on both sides of (3.1) (Resp. (3.2)). Thus, in general x and y cannot be solved directly via (3.1) and (3.2). The easiest way to remove this difficulty is probably to replace x (Resp. y) in the right-hand-side of (3.1) (Resp. (3.2)) by x^k (Resp. y^k), yielding the following iterative scheme:

$$x^{k+1} = P_{\mathcal{X}}\big[x^k - \beta[f(x^k) - A^T[\lambda^k - H(Ax^k + By^k - b)]]\big], \tag{3.3}$$

and

$$y^{k+1} = P_{\mathcal{Y}}\big[y^k - \beta[g(y^k) - B^T[\lambda^k - H(Ax^k + By^k - b)]]\big]. \tag{3.4}$$

Despite the obvious simplicity, this idea raises immediately the question: How can we ensure the convergence of the sequence generated by (3.3) and (3.4) to a solution of VI (1.1)–(1.3)?

To answer this question, we first show that the formula (3.3) (Resp. (3.4)) is essentially the application of an inexact proximal point algorithm to (3.1) (Resp. (3.2)) with appropriate choices of the proximal parameter and inexact criterion. We only illustrate this fact for (3.3). When the PPA is applied to solve the subproblem (1.4a), we have the following proximally regularized subproblem:

$$x \in \mathcal{X}, \quad (x' - x)^T \left\{ f(x) - A^T[\lambda^k - H(Ax + By^k - b)] + R_k(x - x^k) \right\} \geq 0, \quad \forall\ x' \in \mathcal{X}, \tag{3.5}$$

where $R_k \in \mathcal{R}^{n_1 \times n_1}$ is positive definite. For simplification, we may assume that $R_k \equiv \frac{1}{\beta} \cdot I$ where $\beta > 0$ and I is the identity matrix in $\mathcal{R}^{n_1 \times n_1}$. Then, with this choice of R, the proximally regularized subproblem (3.5) can be rewritten into:

$$x \in \mathcal{X}, \quad (x' - x)^T \left\{ \beta(f(x) - A^T[\lambda^k - H(Ax + By^k - b)]) + (x - x^k) \right\} \geq 0, \quad \forall\ x' \in \mathcal{X}. \tag{3.6}$$

Note that the proximally regularized subproblem (3.6) is still hard to solve, and the practical way of implementing PPA is to consider the inexact version:

$$x \in \mathcal{X}, \quad (x' - x)^T \left\{ \beta(f(x) - A^T[\lambda^k - H(Ax + By^k - b)]) + (x - x^k) + \xi_x^k \right\} \geq 0, \quad \forall\ x' \in \mathcal{X}, \tag{3.7}$$

where $\xi_x^k \in \mathcal{R}^{n_1}$ is an inexact term. We refer to, e.g. [13, 20], for the analysis and choices on the inexact term ξ_x^k. In particular, to alleviate the difficulty of solving the subproblem (3.6), we propose to choose

$$\xi_x^k = \beta(f(x^k) - f(x) + A^T HA(x^k - x)).$$

Then, with this particular ξ_x^k, the subproblem (3.7) reduces to

$$x \in \mathcal{X}, \quad (x' - x)^T \left\{ \beta(f(x^k) - A^T[\lambda^k - H(Ax^k + By^k - b)]) + (x - x^k) \right\} \geq 0, \quad \forall\ x' \in \mathcal{X}. \tag{3.8}$$

By Lemma 2.2, we can easily show that the closed-form solution of (3.8) is given by (3.3).

Similarly, if we apply the inexact PPA to solve (1.4b) and choose the inexact term as

$$\xi_y^k = \beta(g(y^k) - g(y) + B^T HB(y^k - y)),$$

then we can obtain an approximate solution of (1.4b) easily via the explicit formula (3.4).

Thus, we have illustrated that we can obtain approximate solutions of the sub-VIs in (1.4) easily by implementing inexact proximal point algorithms with appropriate proximal parameters and inexact criteria. In the following, we shall investigate the conditions on ξ_x^k and ξ_y^k in order to ensure the convergence of the inexact PSALM when the approximate solutions (3.3) and (3.4) are adopted.

3.2 Algorithm

Now, we present the inexact PSALM for VI (1.1)–(1.3). As we have mentioned, like PSALM in [16], the inexact PSALM is also in the prediction-correction fashion, where the prediction step generates a predictor (denoted by $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ with the given iterate $w^k = (x^k, y^k, \lambda^k)$, and the correction step corrects the predictor to generate the new iterate $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$.

To simplify our following analysis, we denote

$$R_k = r_k I_{n_1}, \qquad S_k = s_k I_{n_2}, \quad \text{and} \quad G_k = \begin{pmatrix} R_k + A^T H A & & \\ & S_k + B^T H B & \\ & & H^{-1} \end{pmatrix}, \tag{3.9}$$

where $r_k > 0$ and $s_k > 0$.

Algorithm Inexact parallel splitting augmented Lagrangian methods for VI (1.1)–(1.3)

Step 0. Let $Q(w)$ be defined in (2.7) and G_k be defined in (3.9). Let $\epsilon > 0$, $\nu \in (0, 1)$, $\gamma \in (0, 2)$ and $H \in \mathcal{R}^{m\times m}$ be positive definite. Let $w^0 = (x^0, y^0, \lambda^0) \in \mathcal{R}^{n_1} \times \mathcal{R}^{n_2} \times \mathcal{R}^m$, $r_0 > 0$, $s_0 > 0$, and $k = 0$.

Step 1. Prediction Step Produce $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ via the following steps:

Step 1.1 Set

$$\tilde{x}^k := P_{\mathcal{X}}\left[x^k - \frac{1}{r_k}\big(f(x^k) - A^T[\lambda^k - H(Ax^k + By^k - b)]\big)\right], \tag{3.10}$$

where $r_k > 0$ is chosen such that

$$\|\xi_x^k\| \le \nu r_k \|x^k - \tilde{x}^k\| \ \text{ with } \ \xi_x^k := f(x^k) - f(\tilde{x}^k) + A^T H A(x^k - \tilde{x}^k). \tag{3.11}$$

Step 1.2 Set

$$\tilde{y}^k := P_{\mathcal{Y}}\left[y^k - \frac{1}{s_k}\big(g(y^k) - B^T[\lambda^k - H(Ax^k + By^k - b)]\big)\right], \tag{3.12}$$

where $s_k > 0$ is chosen such that

$$\|\xi_y^k\| \le \nu s_k \|y^k - \tilde{y}^k\| \ \text{ with } \ \xi_y^k := g(y^k) - g(\tilde{y}^k) + B^T H B(y^k - \tilde{y}^k). \tag{3.13}$$

Step 1.3 Update $\tilde{\lambda}^k$ via

$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b). \tag{3.14}$$

Step 2. Correction Step Generate the new iterate $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ by:

Form I

$$w_{\mathbf{I}}^{k+1} = w^k - \alpha_k d_1(w^k, \tilde{w}^k, \xi^k) \tag{3.15}$$

where

$$d_1(w^k, \tilde{w}^k, \xi^k) = G_k(w^k - \tilde{w}^k) - \xi^k \text{ with } \xi^k = \begin{pmatrix} \xi_x^k \\ \xi_y^k \\ 0 \end{pmatrix}, \tag{3.16}$$

or

Form II

$$w_{\mathbf{II}}^{k+1} = P_{\mathcal{W}}[w^k - \alpha_k d_2(w^k, \tilde{w}^k)], \tag{3.17}$$

where

$$d_2(w^k, \tilde{w}^k) = Q(\tilde{w}^k) + \begin{pmatrix} A^T H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)] \\ B^T H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)] \\ 0 \end{pmatrix}. \tag{3.18}$$

Here, the step size α_k in (3.15) and (3.17) is determined by

$$\alpha_k = \gamma \alpha_k^*, \qquad \alpha_k^* = \frac{\varphi(w^k, \tilde{w}^k, \xi^k)}{\|d_1(w^k, \tilde{w}^k, \xi^k)\|^2}, \tag{3.19}$$

with

$$\begin{aligned} \varphi(w^k, \tilde{w}^k, \xi^k) := (w^k - \tilde{w}^k)^T d_1(w^k, \tilde{w}^k, \xi^k) + (\lambda^k - \tilde{\lambda}^k)^T [A(x^k - \tilde{x}^k) \\ + B(y^k - \tilde{y}^k)]. \end{aligned} \tag{3.20}$$

Remark 3.1 We first illustrate that the conditions (3.11) and (3.13) on the inexact terms are well-defined. In fact, recall that both $f(x)$ and $g(y)$ are assumed to be Lipschitz continuous (see **A.2**). We denote by L_f and L_g the Lipschitz constants of $f(x)$ and $g(y)$, respectively. Obviously, when r_k satisfies

$$r_k \geq \frac{L_f + \|A^T HA\|}{\nu}, \tag{3.21}$$

it follows that

$$\|\xi_x^k\| \overset{(3.11)}{\leq} (L_f + \|A^T HA\|)\|x^k - \tilde{x}^k\| \overset{(3.21)}{\leq} \nu r_k \|x^k - \tilde{x}^k\|,$$

which guarantees the condition (3.11). Analogously, when s_k satisfies

$$s_k \geq \frac{L_g + \|B^T HB\|}{\nu}, \tag{3.22}$$

we have that

$$\|\xi_y^k\| \overset{(3.13)}{\leq} (L_g + \|B^T HB\|)\|y^k - \tilde{y}^k\| \overset{(3.22)}{\leq} \nu s_k \|y^k - \tilde{y}^k\|,$$

which guarantees the condition (3.13). Thus, in the implementation of the proposed algorithm, we can increase the values of r_k and s_k whenever the conditions (3.11) and (3.13) are not satisfied. Note that the Lipschitz continuity of $f(x)$ and $g(y)$ ensures that we can find qualified r_k and s_k in finitely many trails even though the exact values of L_f and L_g are unknown. Therefore, the sequences $\{r_k\}$ and $\{s_k\}$ are both upper bounded. In fact, as we will illustrate in Sect. 5, we can ensure that these sequence are also lower bounded.

Remark 3.2 Because the sequences $\{r_k\}$ and $\{s_k\}$ are both lower and upper bounded, the sequence $\{G_k\}$ defined in (3.9) is also both lower and upper bounded. Thus, we can define

$$\inf_k\{\delta_k \mid \delta_k \text{ is the smallest eigenvalue of the matrix } G_k\} = \delta > 0. \tag{3.23}$$

and

$$\sup_k\{\zeta_k \mid \zeta_k \text{ is the largest eigenvalue of the matrix } G_k\} = \zeta < +\infty. \tag{3.24}$$

Remark 3.3 We note that $\tilde{x}^k$ obtained by (3.10) and $\tilde{y}^k$ obtained by (3.12) are actually solutions of the following VIs:

$$(x' - \tilde{x}^k)^T\{f(x^k) - A^T[\lambda^k - H(Ax^k + By^k - b)] + R_k(\tilde{x}^k - x^k)\} \geq 0,$$
$$\forall\, x' \in \mathcal{X}, \tag{3.25}$$
$$(y' - \tilde{y}^k)^T\{g(y^k) - A^T[\lambda^k - H(Ax^k + By^k - b)] + S_k(\tilde{y}^k - y^k)\} \geq 0,$$
$$\forall\, y' \in \mathcal{Y}. \tag{3.26}$$

With (3.25), (3.26) and (3.14)–(3.18), we have

$$(w' - \tilde{w}^k)^T\left\{d_2(w^k, \tilde{w}^k) - d_1(w^k, \tilde{w}^k, \xi^k)\right\} \geq 0, \quad \forall\, w' \in \mathcal{W}. \tag{3.27}$$

3.3 Relationship to some existing methods

In this subsection, we recall two relevant splitting methods which are applicable for VI (1.1)–(1.3). Then, we delineate their difference from the proposed inexact PSALM. In Sect. 5, we will report their numerical comparison with the proposed inexact PSALM.

We first recall the proximal-based decomposition method (PBDM for abbreviation) proposed in [3]. When this method is applied to solve VI (1.1)–(1.3), the new iterate $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ is generated by solving the following problems:

$$(x' - x^{k+1})^T\left\{f(x^{k+1}) - A^T[\lambda - H(Ax^k + By^k - b) + r_k(x^{k+1} - x^k)]\right\} \geq 0,$$
$$\forall\, x' \in \mathcal{X}, \tag{3.28a}$$
$$(y' - y^{k+1})^T\left\{g(y^{k+1}) - B^T[\lambda - H(Ax^k + By^k - b) + s_k(y^{k+1} - y^k)]\right\} \geq 0,$$
$$\forall\, y' \in \mathcal{Y}, \tag{3.28b}$$
$$\lambda^{k+1} = \lambda^k - H(Ax^{k+1} + By^{k+1} - b),$$

where $r_k > 0$ and $s_k > 0$. Essentially, PBDM also applies the PPA to solve the sub-VIs in (1.4). In fact, in [3], the penalty matrix $H \equiv \beta I$ with $\beta > 0$ and the proximal parameters $r_k \equiv s_k = 1/\beta$. In addition, the value of β is required to satisfy

$$\beta \le \frac{1}{2\max\{\|A\|, \|B\|\}}.$$

Obviously, PBDM is also in the parallel nature since the involved subproblems (3.28a) and (3.28b) can be solved in parallel.

To see the difference of PBDM from the proposed inexact PSALM, we emphasize that the proximally regularized subproblems (3.28a) and (3.28b) are still hard to solve, and they do not have closed-form solutions. Actually, by applying Lemma 2.2 again, we know that the solutions of (3.28a) and (3.28b) are given by:

$$x^{k+1} = P_{\mathcal{X}}\big[x^k - \beta[f(x^{k+1}) - A^T[\lambda^k - H(Ax^k + By^k - b)]]\big] \tag{3.29}$$

and

$$y^{k+1} = P_{\mathcal{Y}}\big[y^k - \beta[g(y^{k+1}) - B^T[\lambda^k - H(Ax^k + By^k - b)]]\big], \tag{3.30}$$

which are both implicit in the sense that their solutions cannot be solved directly. For the proposed inexact PSALM, the resulting sub-VIs are much easier as they have closed-form solutions as given by (3.10) and (3.12). Because of this advantage, we anticipate that the proposed inexact PSALM outperforms PBDM numerically even though it requires additional correction steps, and we will verify this anticipation by some numerical results in Sect. 5.

Another splitting method relevant to the proposed inexact PSALM is the alternating projection based prediction-correction method (APBPCM for abbreviation) proposed in [14]. Like PSALM and the proposed inexact PSALM, APBPCM is also in the prediction-correction fashion and it applies inexact PPA to solve the subproblems. But, APBPCM solves the resulting sub-VIs in the alternating order, which is different from the parallel type methods such as PBDM, PSALM, and the proposed inexact PSALM. More specifically, APBPCM solves the following subproblems to generate the predictor $(\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$:

$$(x' - \tilde{x}^k)^T \big\{f(x^k) - A^T[\lambda - H(Ax^k + By^k - b) + r_k(\tilde{x}^k - x^k)]\big\} \ge 0,$$
$$\forall x' \in \mathcal{X}, \tag{3.31a}$$
$$(y' - \tilde{y}^k)^T \big\{g(y^k) - B^T[\lambda - H(A\tilde{x}^k + By^k - b) + s_k(\tilde{y}^k - y^k)]\big\} \ge 0,$$
$$\forall y' \in \mathcal{Y}, \tag{3.31b}$$
$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b).$$

Then, APBPCM corrects the predictor $(\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ by some correction steps to generate the new iterate $(x^{k+1}, y^{k+1}, \lambda^{k+1})$. Note that the subproblem (3.31b) requires the solution $\tilde{x}^k$ of (3.31a). Thus, these two sub-VIs (3.31a) and (3.31b) must be solved sequentially, rather than in parallel.

4 Convergence

In this section, we establish the convergence of the proposed inexact PSALM. Recall that the approximate solutions (3.3) and (3.4) are derived in the framework of inexact PPA. We first prove a lemma which is very useful in the following analysis.

Lemma 4.1 *Let* $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ *be generated by* (3.10)–(3.14) *from the given* $w^k = (x^k, y^k, \lambda^k)$, *and* $\varphi(w^k, \tilde{w}^k, \xi^k)$ *be defined in* (3.20). *Then, we have*

$$\varphi(w^k, \tilde{w}^k, \xi^k) \geq \min\left\{\frac{2-\sqrt{2}}{2}, 1-\nu\right\}\delta\|w^k - \tilde{w}^k\|^2, \tag{4.1}$$

where δ *is defined in* (3.23).

Proof First, according to the definitions (see (3.9) and (3.20)), we have that

$$\begin{aligned}\varphi(w^k, \tilde{w}^k, \xi^k) = {} & \|A(x^k - \tilde{x}^k)\|_H^2 + \|B(y^k - \tilde{y}^k)\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 + \|x^k - \tilde{x}^k\|_{R_k}^2 \\ & + \|y^k - \tilde{y}^k\|_{S_k}^2 + (Ax^k - A\tilde{x}^k)^T(\lambda^k - \tilde{\lambda}^k) \\ & + (By^k - B\tilde{y}^k)^T(\lambda^k - \tilde{\lambda}^k) - (w^k - \tilde{w}^k)^T\xi^k. \end{aligned} \tag{4.2}$$

By using the Cauchy-Schwarz inequality, we get

$$(Ax^k - A\tilde{x}^k)^T(\lambda^k - \tilde{\lambda}^k) \geq -\frac{\sqrt{2}}{2}\|A(x^k - \tilde{x}^k)\|_H^2 - \frac{\sqrt{2}}{4}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2, \tag{4.3}$$

and

$$(By^k - B\tilde{y}^k)^T(\lambda^k - \tilde{\lambda}^k) \geq -\frac{\sqrt{2}}{2}\|B(y^k - \tilde{y}^k)\|_H^2 - \frac{\sqrt{2}}{4}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2. \tag{4.4}$$

Using (3.11), (3.13) and the Cauchy-Schwarz inequality, we have that

$$(w^k - \tilde{w}^k)^T\xi^k \geq -\nu\begin{pmatrix} x^k - \tilde{x}^k \\ y^k - \tilde{y}^k \end{pmatrix}^T \begin{pmatrix} R_k & 0 \\ 0 & S_k \end{pmatrix}\begin{pmatrix} x^k - \tilde{x}^k \\ y^k - \tilde{y}^k \end{pmatrix}.$$

Substituting the above three inequalities into (4.2), we obtain

$$\begin{aligned}\varphi(w^k, \tilde{w}^k) &\geq \min\left\{\frac{2-\sqrt{2}}{2}, 1-\nu\right\}\|w^k - \tilde{w}^k\|_{G_k}^2 \\ &\geq \min\left\{\frac{2-\sqrt{2}}{2}, 1-\nu\right\}\lambda_{min}(G_k)\|w^k - \tilde{w}^k\|^2 \qquad (4.5) \\ &\geq \min\left\{\frac{2-\sqrt{2}}{2}, 1-\nu\right\}\delta\|w^k - \tilde{w}^k\|^2, \qquad (4.6)\end{aligned}$$

which completes the proof. □

Before we prove the convergence of the proposed inexact PSALM, we first illustrate why we choose the step size α_k as (3.19). Let $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$ be an arbitrary solution point of VI$(Q, \mathcal{W})$. In order to find an appropriate step size α for the correction step, we denote by $w_I^{k+1}(\alpha)$ and $w_{II}^{k+1}(\alpha)$ the correction forms I and II with a undetermined step size, i.e.,

$$w_{\mathbf{I}}^{k+1}(\alpha) := w^k - \alpha d_1(w^k, \tilde{w}^k, \xi^k), \tag{4.7}$$

and

$$w_{\mathbf{II}}^{k+1}(\alpha) := P_{\mathcal{W}}[w^k - \alpha d_2(w^k, \tilde{w}^k)], \tag{4.8}$$

where $\alpha > 0$ is the step size to be determined. Moreover, we measure the improvement obtained by the $(k+1)$-th iteration by:

$$\Theta_k(\alpha) := \|w^k - w^*\|^2 - \|w^{k+1}(\alpha) - w^*\|^2, \tag{4.9}$$

which is dependent on the value of the undetermined step size α. Our natural desire is to find such an optimal value of α that maximizes the function $\Theta_k(\alpha)$ at each iteration. Unfortunately, due to the lack of w^*, this goal is not practical. Therefore, we do the next-best thing: find a lower bound of $\Theta_k(\alpha)$ which does not involve w^*, and then choose a value of α to maximize this lower bound. This objective is realized in the following theorem.

Theorem 4.1 *Let $w^{k+1}(\alpha)$ be the correction step* (4.7) *or* (4.8) *with a undetermined step size, and $\Theta_k(\alpha)$ be defined in* (4.9). *Then we have*

$$\Theta_k(\alpha) \geq \Psi_k(\alpha), \tag{4.10}$$

with

$$\Psi_k(\alpha) := 2\alpha\varphi(w^k, \tilde{w}^k, \xi^k) - \alpha^2\|d_1(w^k, \tilde{w}^k, \xi^k)\|^2, \tag{4.11}$$

where $\varphi(w^k, \tilde{w}^k, \xi^k)$ and $d_1(w^k, \tilde{w}^k, \xi^k)$ are defined in (3.20) *and* (3.16), *respectively.*

Proof We divide the proof into two parts for (4.7) and (4.8), respectively.

(I). We first prove the assertion for the correction form (4.7). For this purpose, we need to prove the following inequality:

$$(w^k - w^*)^T d_1(w^k, \tilde{w}^k, \xi^k) \geq \varphi(w^k, \tilde{w}^k, \xi^k). \tag{4.12}$$

In fact, by using Lemma 2.2, we can reformulate (3.27) into:

$$\tilde{w}^k = P_{\mathcal{W}}\big[\tilde{w}^k - [d_2(w^k, \tilde{w}^k) - d_1(w^k, \tilde{w}^k, \xi^k)]\big]. \tag{4.13}$$

Setting $v = \tilde{w}^k - (d_2(w^k, \tilde{w}^k) - d_1(w^k, \tilde{w}^k, \xi^k))$ and $u = w^*$ in (2.1), we get that

$$(w^* - \tilde{w}^k)^T\big\{[\tilde{w}^k - (d_2(w^k, \tilde{w}^k) - d_1(w^k, \tilde{w}^k, \xi^k))] - \tilde{w}^k\big\} \leq 0,$$

that is,

$$(w^* - \tilde{w}^k)^T (d_2(w^k, \tilde{w}^k) - d_1(w^k, \tilde{w}^k, \xi^k)) \geq 0.$$

Then, it follows that

$$(\tilde{w}^k - w^*)^T d_1(w^k, \tilde{w}^k, \xi^k) \geq (\tilde{w}^k - w^*)^T d_2(w^k, \tilde{w}^k). \tag{4.14}$$

On the other hand, using the definition and monotonicity of $Q(w)$ (see (2.7)), we have

$$\begin{aligned}
&(\tilde{w}^k - w^*)^T d_2(w^k, \tilde{w}^k) \\
&\quad = (\tilde{w}^k - w^*)^T \left\{ Q(\tilde{w}^k) + \begin{pmatrix} A^T H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)] \\ B^T H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)] \\ 0 \end{pmatrix} \right\} \\
&\quad \geq (\tilde{w}^k - w^*) \begin{pmatrix} A^T H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)] \\ B^T H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)] \\ 0 \end{pmatrix} \\
&\quad = (\lambda^k - \tilde{\lambda}^k)^T [A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)],
\end{aligned} \tag{4.15}$$

where the inequality follows from the monotonicity of $Q(w)$; and the last equality is because $Ax^* + By^* - b = 0$ and

$$A\tilde{x}^k + B\tilde{y}^k - b = H^{-1}(\lambda^k - \tilde{\lambda}^k),$$

which is derived from (3.14). Note that $w^* \in \mathcal{W}^*$, we thus have that

$$(\tilde{w}^k - w^*)^T Q(\tilde{w}^k) \geq (\tilde{w}^k - w^*)^T Q(w^*) \geq 0.$$

Using the definition of $\varphi(w^k, \tilde{w}^k, \xi^k)$ (see (3.20)), we have that

$$(\tilde{w}^k - w^*)^T d_2(w^k, \tilde{w}^k) \geq \varphi(w^k, \tilde{w}^k) - (w^k - \tilde{w}^k)^T d_1(w^k, \tilde{w}^k).$$

Then, the inequality (4.12) follows immediately from the above inequality and (4.14).

By a straightforward manipulation, we have that

$$\begin{aligned}
\Theta_k(\alpha) &\overset{\text{def}}{=} \|w^k - w^*\|^2 - \|w^{k+1}(\alpha) - w^*\|^2 \\
&\overset{(4.7)}{=} \|w^k - w^*\|^2 - \|w^k - \alpha d_1(w^k, \tilde{w}^k, \xi^k) - w^*\|^2 \\
&= 2\alpha(w^k - w^*)^T d_1(w^k, \tilde{w}^k, \xi^k) - \alpha^2 \|d_1(w^k, \tilde{w}^k, \xi^k)\|^2 \\
&\geq 2\alpha\varphi(w^k, \tilde{w}^k, \xi^k) - \alpha^2 \|d_1(w^k, \tilde{w}^k, \xi^k)\|^2 \\
&\overset{(4.11)}{=} \Psi_k(\alpha),
\end{aligned} \tag{4.16}$$

which implies that the assertion (4.10) holds for the correction form (4.7).

(II). Now, we prove the assertion (4.10) for the second correction form (4.8). At the first stage, it follows from (4.15) that

$$(w^k - w^*)^T d_2(w^k, \tilde{w}^k) \geq (w^k - \tilde{w}^k)^T d_2(w^k, \tilde{w}^k) + (\lambda^k - \tilde{\lambda}^k)^T [A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)]. \tag{4.17}$$

Then, since $w^* \in \mathcal{W}$ and $w^{k+1}(\alpha) = P_{\mathcal{W}}[w^k - \alpha d_2(w^k, \tilde{w}^k)]$, it follows from (2.3) that

$$\|w^{k+1}(\alpha) - w^*\|^2 \leq \|w^k - \alpha d_2(w^k, \tilde{w}^k) - w^*\|^2 - \|w^k - \alpha d_2(w^k, \tilde{w}^k) - w^{k+1}(\alpha)\|^2.$$

Consequently, we get

$$\begin{aligned} \Theta_k(\alpha) &= \|w^k - w^*\|^2 - \|w^{k+1}(\alpha) - w^*\|^2 \\ &\overset{(2.3)}{\geq} \|w^k - w^*\|^2 + \|w^k - w^{k+1}(\alpha) - \alpha d_2(w^k, \tilde{w}^k)\|^2 - \|w^k - w^* \\ &\quad - \alpha d_2(w^k, \tilde{w}^k)\|^2 \\ &= \|w^k - w^{k+1}(\alpha)\|^2 + 2\alpha\{w^{k+1}(\alpha) - w^k\}^T d_2(w^k, \tilde{w}^k) \\ &\quad + 2\alpha(w^k - w^*)^T d_2(w^k, \tilde{w}^k). \end{aligned} \tag{4.18}$$

Applying (4.17) to the last term of the right-hand-side of (4.18), we obtain

$$\begin{aligned} \Theta_k(\alpha) \geq{}& \|w^k - w^{k+1}(\alpha)\|^2 + 2\alpha\{w^{k+1}(\alpha) - \tilde{w}^k\}^T d_2(w^k, \tilde{w}^k) \\ &+ 2\alpha(\lambda^k - \tilde{\lambda}^k)^T [A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)]. \end{aligned} \tag{4.19}$$

Since $w^{k+1}(\alpha) \in \mathcal{W}$ and (4.13), it follows from (2.1) that

$$\begin{aligned} &0 \geq 2\alpha(w^{k+1}(\alpha) - \tilde{w}^k)^T \{[\tilde{w}^k - (d_2(w^k, \tilde{w}^k) - d_1(w^k, \tilde{w}^k))] - \tilde{w}^k\}, \\ &\forall \alpha > 0. \end{aligned} \tag{4.20}$$

Adding (4.19) and (4.20) together, we get

$$\begin{aligned} \Theta_k(\alpha) \geq{}& \|w^k - w^{k+1}(\alpha)\|^2 + 2\alpha\{w^{k+1}(\alpha) - \tilde{w}^k\}^T d_1(w^k, \tilde{w}^k, \xi^k) \\ &+ 2\alpha(\lambda^k - \tilde{\lambda}^k)^T [A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)]. \end{aligned}$$

By regrouping the right-hand-side of the above inequality, we obtain

$$\begin{aligned} \Theta_k(\alpha) &\geq \|(w^k - w^{k+1}(\alpha)) - \alpha d_1(w^k, \tilde{w}^k, \xi^k)\|^2 - \alpha^2 \|d_1(w^k, \tilde{w}^k, \xi^k)\|^2 \\ &\quad + 2\alpha\{(\lambda^k - \tilde{\lambda}^k)^T [A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)] \\ &\quad + (w^k - \tilde{w}^k)^T d_1(w^k, \tilde{w}^k, \xi^k)\} \\ &\overset{(3.20)}{\geq} 2\alpha\varphi(w^k, \tilde{w}^k, \xi^k) - \alpha^2 \|d_1(w^k, \tilde{w}^k, \xi^k)\|^2, \end{aligned}$$

which indicates the validity of the assertion (4.10) for the second correction step (4.8). The proof is completed. □

Now, the reason for choosing the step size α_k as (3.19) is clear. In fact, based on the result in Theorem 4.1, we know that

$$\|w^{k+1}(\alpha) - w^*\|^2 \le \|w^k - w^*\|^2 - \Psi_k(\alpha), \tag{4.21}$$

which motivates us to seek such a value of α that maximizes the lower bound $\Psi_k(\alpha)$ at each iteration. Since $\Psi_k(\alpha)$ is a quadratic function of α (see (4.11)), it reaches its maximum at

$$\alpha_k^* = \frac{\varphi(w^k, \tilde{w}^k, \xi^k)}{\|d_1(w^k, \tilde{w}^k, \xi^k)\|^2}. \tag{4.22}$$

Note that we also attach a relaxation factor γ to the second-best step size (4.22) because we only maximize the lower bound $\Psi_k(\alpha)$, rather than $\Theta_k(\alpha)$.

By setting the step size in the correction step (3.15) or (3.17) as $\alpha_k = \gamma\alpha_k^*$, we can easily derive that

$$\begin{aligned}\Psi_k(\alpha_k) = \Psi_k(\gamma\alpha_k^*) &\overset{(4.11)}{=} 2\gamma\alpha_k^*\varphi(w^k, \tilde{w}^k, \xi^k) - (\gamma^2\alpha_k^*)(\alpha_k^*\|d_1(w^k, \tilde{w}^k, \xi^k)\|^2) \\ &\overset{(3.19)}{=} \gamma(2-\gamma)\alpha_k^*\varphi(w^k, \tilde{w}^k, \xi^k). \end{aligned} \tag{4.23}$$

Since $\varphi(w^k, \tilde{w}^k, \xi^k) > 0$ whenever a solution is not found (see Lemma 4.1), it follows from (4.21) that the relaxation factor must satisfy $\gamma \in (0, 2)$ for the purpose of generating a new iterate which is closer to the solution set, i.e., $\Psi_k(\alpha_k) > 0$.

Based on the above analysis, we instantly have the following corollary of Theorem 4.1.

Corollary 4.1 *Let the sequence $\{w^k\}$ be generated by the proposed inexact PSALM and $w^* \in \mathcal{W}^*$. Then, we have*

$$\|w^{k+1} - w^*\|^2 \le \|w^k - w^*\|^2 - \gamma(2-\gamma)\alpha_k^*\varphi(w^k, \tilde{w}^k, \xi^k). \tag{4.24}$$

The next lemma indicates that α_k^* defined in (4.22) is uniformly bounded below from a positive number for all iterations generated by the proposed PSALM.

Lemma 4.2 *For the proposed inexact PSALM, there exists a constant $c_1 > 0$ such that $\alpha_k^* \ge c_1$ for any $k > 0$.*

Proof First, it follows from (3.11) and (3.13) that

$$\begin{aligned}\|\xi^k\|^2 &= \|\xi_x^k\|^2 + \|\xi_y^k\|^2 \\ &\overset{(3.11,\,3.13)}{\le} \nu^2(r_k^2\|x^k - \tilde{x}^k\|^2 + s_k^2\|y^k - \tilde{y}^k\|^2) \\ &\le \nu^2\|G_k\|^2\|w^k - \tilde{w}^k\|^2 \\ &\le \nu^2\zeta^2\|w^k - \tilde{w}^k\|^2, \end{aligned} \tag{4.25}$$

where the last inequality comes from (3.24). Therefore, we derive that

$$\begin{aligned}\|d_1(w^k,\tilde{w}^k,\xi^k)\| &= \|G_k(w^k-\tilde{w}^k)-\xi^k\| \\ &\le \|G_k\|\|w^k-\tilde{w}^k\|+\|\xi^k\| \\ &\overset{(4.25)}{\le} (\|G_k\|+\nu\zeta)\|w^k-\tilde{w}^k\| \\ &\le (1+\nu)\zeta\|w^k-\tilde{w}^k\|, \end{aligned} \tag{4.26}$$

where the last inequality is again because of (3.24).

Thus, according to (4.1) and (4.26), we have that

$$\alpha_k^* = \frac{\varphi(w^k,\tilde{w}^k,\xi^k)}{\|d_1(w^k,\tilde{w}^k,\xi^k)\|^2} \ge c_1 := \min\left\{\frac{2-\sqrt{2}}{2}, 1-\nu\right\}\frac{\delta}{(1+\nu)^2\zeta^2} > 0,$$

which is the assertion. □

Now, based on Lemma 4.1, Corollary 4.1 and Lemma 4.2, we have the following corollary from which we can establish the convergence of the proposed inexact PSALM easily.

Corollary 4.2 *Let $\{w^k\}$ be the sequence generated by the proposed inexact PSALM and $w^* \in \mathcal{W}^*$. Then, we have*

$$\|w^{k+1}-w^*\|^2 \le \|w^k-w^*\|^2 - \gamma(2-\gamma)\min\left\{\frac{2-\sqrt{2}}{2}, 1-\nu\right\}\delta c_1\|w^k-\tilde{w}^k\|^2. \tag{4.27}$$

We need the following result to prove the convergence of the proposed inexact PSALM.

Lemma 4.3 *Let $\{w^k\}$ and $\{\tilde{w}^k\}$ be generated by the proposed inexact PSALM. Let $E_{[\mathcal{W},Q]}(w)$ be defined in (2.8). Then, there exists a constant $c_2 > 0$ such that*

$$\|E_{[\mathcal{W},Q]}(\tilde{w}^k)\| \le c_2\|w^k-\tilde{w}^k\|, \quad \forall k > 0. \tag{4.28}$$

Proof Using Lemma 2.2, (3.16), (3.18) and (4.13), we get

$$\begin{aligned}\tilde{w}^k = P_{\mathcal{W}}\big\{&\tilde{w}^k - \{Q(\tilde{w}^k)+(A,B,0)^T H[A(x^k-\tilde{x}^k)+B(y^k-\tilde{y}^k)] \\ &- G_k(w^k-\tilde{w}^k)+\xi^k\}\big\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|E_{[\mathcal{W},Q]}(\tilde{w}^k)\| &= \Big\|P_{\mathcal{W}}\big\{\tilde{w}^k-\{Q(\tilde{w}^k)+(A,B,0)^T H[A(x^k-\tilde{x}^k) \\ &\quad + B(y^k-\tilde{y}^k)]-G_k(w^k-\tilde{w}^k)+\xi^k\}\big\} - P_{\mathcal{W}}\{\tilde{w}^k-Q(\tilde{w}^k)\}\Big\| \\ &\le \|G_k(w^k-\tilde{w}^k)-\xi^k-(A,B,0)^T H[A(x^k-\tilde{x}^k)\end{aligned}$$

$$
\begin{aligned}
&+ B(y^k - \tilde{y}^k)]\| \\
\le\ & \|G_k\|\|w^k - \tilde{w}^k\| + \|(A, B, 0)^T H[A(x^k - \tilde{x}^k) \\
&+ B(y^k - \tilde{y}^k)]\| + \|\xi^k\| \\
\overset{(4.25)}{\le}\ & \zeta(1+\nu)\|w^k - \tilde{w}^k\| + \|(A, B, 0)^T H[A(x^k - \tilde{x}^k) \\
&+ B(y^k - \tilde{y}^k)]\|,
\end{aligned}
$$

where the first inequality comes from (2.2). Hence, the assertion is proved. □

Now, we are ready to establish the convergence of the proposed inexact PSALM.

Theorem 4.2 *The sequence $\{w^k\}$ generated by the proposed inexact PSALM converges to some $w^\infty \in \mathcal{W}^*$.*

Proof First, it follows from Corollary 4.2 that

$$\|w^{k+1} - w^*\|^2 \le \|w^k - w^*\|^2, \tag{4.29}$$

which implies that the sequence $\{w^k\}$ is bounded. Moreover, since $\nu \in (0, 1)$ and $\gamma \in (0, 2)$, it follows from (4.27) that

$$\gamma(2-\gamma)\min\left\{\frac{2-\sqrt{2}}{2}, 1-\nu\right\}\delta c_1 \sum_{k=0}^{\infty} \|w^k - \tilde{w}^k\|^2 \le \|w^0 - w^*\|^2,$$

which implies that $\lim_{k\to\infty} \|w^k - \tilde{w}^k\|^2 = 0$. We thus conclude that the sequence $\{\tilde{w}^k\}$ is also bounded. In addition, according to Lemma 4.3, we have that $\lim_{k\to\infty} \|E_{[\mathcal{W}, Q]}(\tilde{w}^k)\| = 0$.

Let w^∞ be a cluster point of $\{\tilde{w}^k\}$ and $\{\tilde{w}^{k_j}\}$ be the subsequence converging to w^∞. Since $E_{[\mathcal{W}, Q]}(\tilde{w}^k)$ is a continuous function of w, it follows that $E_{[\mathcal{W}, Q]}(w^\infty) = \lim_{j\to\infty} E_{[\mathcal{W}, Q]}(\tilde{w}^{k_j}) = 0$. Then, from Lemma 2.3, we know that $w^\infty \in \mathcal{W}^*$, i.e., w^∞ is a solution of VI($\mathcal{W}$, Q).

Now, we have to show that the sequence $\{w^k\}$ actually converges to w^∞. Since $\lim_{k\to\infty} \|\tilde{w}^k - w^k\| = 0$ and $\{\tilde{w}^{k_j}\} \to w^\infty$, for any given $\varepsilon > 0$, there exists $l > 0$, such that

$$\|w^{k_l} - \tilde{w}^{k_l}\| < \frac{\varepsilon}{2}, \quad \text{and} \quad \|\tilde{w}^{k_l} - w^\infty\| < \frac{\varepsilon}{2}. \tag{4.30}$$

Therefore, for any $k \ge k_l$, it follows from (4.29) and (4.30) that $\|w^k - w^\infty\| \le \varepsilon$. Thus, the sequence $\{w^k\}$ converges to $w^\infty \in \mathcal{W}^*$. The proof is completed. □

5 Numerical experiments

In this section, we apply the proposed inexact PSALM to solve some applications of the VI (1.1)–(1.3) arising in traffic equilibrium problems, and thus show its efficiency by preliminary numerical results. In particular, we will compare the inexact PSALM numerically with PBDM in [3], APBPCM in [14] and PSALM in [16]. All codes were written by MATLAB v7.8.0 (R2009a) and all numerical experiments were done on a T6500 laptop with CPU 2.6 GHz and 1.75 GB memory.

5.1 Implementation details

As we have mentioned, a critical technique to implement the proposed inexact PSALM is to choose appropriate r_k and s_k to satisfy the conditions (3.11) and (3.13). Let

$$\nu_k^x := \|\xi_x^k\| / \big(r_k \|x^k - \tilde{x}^k\|\big),$$

and

$$\nu_k^y := \|\xi_y^k\| / \big(s_k \|y^k - \tilde{y}^k\|\big).$$

Then, empirically we update the values of r_k and s_k subject to the following rules:

$$r_{k+1} := \begin{cases} r_k * \kappa, & \text{if } \nu_k^x > \nu, \\ r_k, & \text{otherwise,} \end{cases}$$

and

$$s_{k+1} := \begin{cases} s_k * \kappa, & \text{if } \nu_k^y > \nu, \\ s_k & \text{otherwise,} \end{cases}$$

where $\kappa > 1$ is a scaling constant. Therefore, we can easily ensure that the sequences $\{r_k\}$ and $\{s_k\}$ are lower bounded in the implementation of the proposed inexact PSALM. On the other hand, recall that the Lipschitz continuity of $f(x)$ and $g(y)$ ensures that the conditions (3.11) and (3.13) can be satisfied by increasing the values of r_k and s_k in finitely many times. Thus, the sequences $\{r_k\}$ and $\{s_k\}$ are also upper bounded.

In the following, we delineate the implementation details of the proposed inexact PSALM with the second correction form (3.17), and we omit the delineation for the first correction form (3.15) for the sake of succinctness.

Implementation details of the proposed inexact PSALM

Step 0. Let $\epsilon > 0$, $\nu \in (0.5, 1)$, $\gamma \in (0, 2)$, $\kappa > 1$, $r_{min} > 0$, $s_{min} > 0$ and $H \in \mathcal{R}^{m\times m}$ be positive definite. Let $w^0 = (x^0, y^0, \lambda^0) \in \mathcal{R}^{n_1} \times \mathcal{R}^{n_2} \times \mathcal{R}^m$, $r_0 > r_{min}$, $s_0 > s_{min}$ and $k = 0$.

Step 1. Prediction Step Produce $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ via the following steps:

Step 1.1. Calculate $\tilde{x}^k$:

1) Set $p_x^k := f(x^k) - A^T[\lambda^k - H(Ax^k + By^k - b)]$.
2) $\tilde{x}^k := P_{\mathcal{X}}[x^k - p_x^k / r_k]$;
   $\xi_x^k := f(x^k) - f(\tilde{x}^k) + A^T HA(x^k - \tilde{x}^k)$;
   $\nu_k^x := \|\xi_x^k\| / (r_k \|x^k - \tilde{x}^k\|)$.
3) If $\nu_k^x > \nu$, then increase r_k by $\quad r_k := r_k * \nu_k^x * \kappa \quad$ and go to 2).

Step 1.2. Calculate $\tilde{y}^k$:

1) Set $p_y^k := g(y^k) - B^T[\lambda^k - H(Ax^k + By^k - b)]$.

2) $\tilde{y}^k := P_{\mathcal{Y}}[y^k - p_y^k/s_k]$;
$\xi_y^k := g(y^k) - g(\tilde{y}^k) + B^T HB(y^k - \tilde{y}^k)$;
$\nu_k^y := \|\xi_y^k\|/(s_k\|y^k - \tilde{y}^k\|)$.

3) If $\nu_k^y > \nu$, then increase s_k by $s_k := s_k * \nu_k^y * \kappa$ and go to 2).

Step 1.3. Calculate $\tilde{\lambda}^k$: set $p_\lambda^k := H(A\tilde{x}^k + B\tilde{y}^k - b)$;
$\tilde{\lambda}^k := \lambda^k - p_\lambda^k$.

Step 2. Calculate the direction of the correction step (3.18):
Set $q_x^k := f(\tilde{x}^k) - A^T\tilde{\lambda}^k + A^T H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)]$;
$q_y^k := g(\tilde{y}^k) - B^T\tilde{\lambda}^k + B^T H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)]$.

Step 3. Calculate the step-size in the correction step: $\alpha_k = \gamma\alpha_k^*$, (the formula of α_k^* see (3.19)).

Step 4. Correction Step *Generate the new iterate $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ by the following step:*
$x^{k+1} = P_{\mathcal{X}}\{x^k - \alpha_k q_x^k\}$;
$y^{k+1} = P_{\mathcal{Y}}\{y^k - \alpha_k q_y^k\}$;
$\lambda^{k+1} = \lambda^k - \alpha_k p_\lambda^k$;

Step 5. Adjust the values of r_k and s_k:

$$r_{k+1} := \begin{cases} \max\{r_{min},\ r_k * \nu_k^x * \kappa\}, & \text{if } \nu_k^x \le 0.5, \\ r_k, & \text{otherwise.} \end{cases}$$

$$s_{k+1} := \begin{cases} \max\{s_{min},\ s_k * \nu_k^y * \kappa\}, & \text{if } \nu_k^y \le 0.5, \\ s_k. & \text{otherwise.} \end{cases}$$

$k := k + 1$, go to Step 1.

5.2 Numerical results for traffic equilibrium problems

In this subsection, we test some specific applications of the VI (1.1)–(1.3) arising in traffic equilibrium problems with link capacity bounds, which have been well studied in the literature of transportation. In particular, we test Examples 7.4 and 7.5 in [19], and we refer to [15] for the procedure of reformulating these traffic equilibrium problems into VIs. Overall, these traffic equilibrium problems with link capacity bounds can be characterized by VIs with linear inequality constraints:

$$(x - x^*)^T f(x^*) \ge 0, \quad \forall\, x \in \Pi, \tag{5.1}$$

with

$$\Pi = \{x \in R^n \mid A^T x \le b,\ \ x \ge 0\}, \tag{5.2}$$

where $x \in \mathcal{R}^n$ represents the traffic flow on paths, b is the vector indicating the capacities on links, $A \in \mathcal{R}^{n\times m}$ is the path-link indicating matrix, and f is the vector indicating the traffic flows on links, see [15] for details. Obviously, by introducing the slack variable $y \ge 0$, VI (5.1)–(5.2) is equivalent to

$$(x - x^*)^T f(x^*) \ge 0, \quad \forall\, x \in \Omega, \tag{5.3}$$

with

$$\Omega = \{(x,y) | A^T x + y = b,\ \ x \geq 0,\ \ y \geq 0\}, \tag{5.4}$$

which is a special case of VI (1.1)–(1.3) with $g(y) \equiv 0$, $B = I$, $\mathcal{X} = \mathcal{R}^n_+$ and $\mathcal{Y} = \mathcal{R}^m_+$. For Example 7.4 in [19], $n = 49$, $m = 28$ and $A \in \mathcal{R}^{49\times 28}$; and for Example 7.5 in [19], $n = 55$, $m = 37$ and $A \in \mathcal{R}^{55\times 37}$.

In the implementation of the proposed inexact PSALM, we set $\nu = 0.95$, $\gamma = 1.85$, $\kappa = 1.25$, $r_0 = 1$, $s_0 = 1.1$, $H = \beta I$ with $\beta = 1.1$ and the initial iterative is $x^0 = \mathbf{1}$ and $y^0 = \lambda^0 = \mathbf{0}$. For the parameters of APBPCM, we take their values as recommended in [14]. For the parameters of PSALM, the values are taken as the proposed inexact PSALM. Note that both PBDM and PSALM apply the method in [12] to solve the resulting sub-VIs iteratively. We use the following stopping criterion:

$$\max\left\{\frac{\|e_x(w^k)\|_\infty}{\|e_x(w^0)\|_\infty}, \|e_y(w^k)\|_\infty, \|e_\lambda(w^k)\|_\infty\right\} \leq \varepsilon. \tag{5.5}$$

In Table 1, we report the numerical performance for PBDM, APBPCM, PSALM and the proposed inexact PSALM (denoted by "IPSALM") for Examples 7.4 and 7.5 in [19] with different values of b (link capacity). We report the number of iterations (No. of iterations), the number of function evaluations (No. of F evaluations), and the CPU time in seconds (Time). Since PBDM and PSALM need to solve the resulting sub-VIs iteratively at the inner loops, we report the aggregate numbers of inner iterations in the parenthesis for these two methods.

The data in Table 1 illustrates the efficiency of the proposed inexact PSALM and its superiority to PBDM, APBPCM and PSALM in terms of both CPU time and number of function evaluations.

As illustrated in [15, 19], the solution of VI (5.3)–(5.4) indicates the flow on all paths under consideration, and the Lagrange multiplier λ^* actually means the toll that should be charged on links to avoid congestion. To see the traffic flows and toll on links when the equilibrium is achieved, we report their values generated by the proposed inexact PSALM in Tables 2 and 3, respectively, for the tested examples with the capacity $b = 40$.

Note that the data in Tables 2 and 3 indicates that no toll is charged for those links whose flows are lower than their capacities.

6 Conclusions

In this paper, we proposed the inexact parallel splitting augmented Lagrangian method (PSALM) for variational inequalities with separable structures. The new method improves the exact PSALM in [16] in the sense that the resulting subproblems can be solved easily with closed-form solutions. We verify the efficiency of the new method by some traffic equilibrium problems.

In [16], PSALM was extended to solve VI (1.1) with three separable blocks:

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad F(u) = \begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}, \tag{6.1}$$

Table 1 Numerical comparison of PBDM, APBPCM, PSALM and IPSALM for various ε

Examples	b	No. of iterations				No. of F evaluations				Time			
		PBDM	APBPCM	PSALM	IPSALM	PBDM	APBPCM	PSALM	IPSALM	PBDM	APBPCM	PSALM	IPSALM
$\varepsilon = 10^{-4}$													
Ex. 7.4 in [19]	30	928(4632)	300	97(25191)	193	10354	634	22702	403	0.43	0.10	0.80	0.07
	40	949(4763)	336	82(27298)	190	10736	714	16110	409	0.38	0.11	0.60	0.07
Ex. 7.5 in [19]	30	1104(7304)	318	149(66748)	382	15889	680	25474	783	0.43	0.11	1.00	0.14
	40	1128(8947)	476	158(70819)	426	19441	1024	25284	883	0.50	0.15	0.99	0.15
$\varepsilon = 10^{-5}$													
Ex. 7.4 in [19]	30	1134(4869)	384	104(25198)	255	13624	802	23458	530	0.43	0.11	0.82	0.09
	40	1155(5015)	415	152(27368)	247	12717	872	17146	520	0.43	0.14	0.64	0.10
Ex. 7.5 in [19]	30	1352(7552)	401	250(66855)	428	11692	862	26702	875	0.51	0.12	1.07	0.12
	40	1375(9194)	593	239(70912)	467	13523	1284	26696	956	0.55	0.15	1.06	0.14
$\varepsilon = 10^{-6}$													
Ex. 7.4 in [19]	30	1340(5075)	467	156(25250)	262	14242	968	24458	544	0.45	0.13	0.87	0.10
	40	1361(5221)	495	270(27486)	269	13335	1032	18158	560	0.42	0.14	0.69	0.10
Ex. 7.5 in [19]	30	1600(7778)	480	546(67151)	466	12436	1028	28382	967	0.53	0.14	1.15	0.14
	40	1623(9442)	722	566(71239)	530	14268	1570	28056	1082	0.58	0.17	1.15	0.16

Table 2 Flow and toll for Example 7.4 in [19] with $b = 40$

Link	Flow	Charge	Link	Flow	Charge	Link	Flow	Charge	Link	Flow	Charge
1	0	0	8	32.90	0	15	27.06	0	22	33.95	0
2	12.94	0	9	0	0	16	5.27	0	23	0	0
3	40.00	25.2	10	0	0	17	1.83	0	24	12.94	0
4	12.94	0	11	0	0	18	32.90	0	25	40.00	124.6
5	0	0	12	33.95	0	19	0	0	26	32.33	0
6	40.00	125.4	13	27.06	0	20	0	0	27	34.16	0
7	34.73	0	14	12.94	0	21	0	0	28	0	0

Table 3 Flow and toll for Example 7.5 in [19] with $b = 40$

Link	Flow	Charge	Link	Flow	Charge	Link	Flow	Charge	Link	Flow	Charge
1	40.00	4.3	11	1.85	0	21	40.00	1.1	31	11.96	0
2	38.15	0	12	11.96	0	22	40.00	136.6	32	40.00	164.2
3	40.00	163.2	13	26.19	0	23	26.19	0	33	40.00	135.7
4	13.81	0	14	13.81	0	24	0	0	34	26.19	0
5	0	0	15	0	0	25	0	0	35	28.04	0
6	0	0	16	0	0	26	0	0	36	40.00	301.3
7	0	0	17	0	0	27	0	0	37	0	0
8	0	0	18	0	0	28	0	0	–	–	–
9	0	0	19	0	0	29	26.19	0	–	–	–
10	40.00	1.1	20	40.00	1.8	30	1.85	0	–	–	–

and

$$\Omega = \{(x, y, z) | \ Ax + By + Cz = b, \ x \in \mathcal{X}, \ y \in \mathcal{Y}, \ z \in \mathcal{Z}\}, \tag{6.2}$$

where $\mathcal{X} \subset \mathcal{R}^{n_1}$, $\mathcal{Y} \subset \mathcal{R}^{n_2}$ and $\mathcal{Z} \subset \mathcal{R}^{n_3}$ are nonempty closed and convex sets; $A \in \mathcal{R}^{m \times n_1}$, $B \in \mathcal{R}^{m \times n_2}$ and $C \in \mathcal{R}^{m \times n_3}$ are given matrices; $f : \mathcal{X} \to \mathcal{R}^{n_1}$, $g : \mathcal{Y} \to \mathcal{R}^{n_2}$ and $h : \mathcal{Y} \to \mathcal{R}^{n_3}$ are given monotone and Lipschitz continuous mappings; $b \in \mathcal{R}^m$ is a given vector and $n_1 + n_2 + n_3 = n$. Without any difficulty, the proposed inexact PSALM can also be extended to solve VI(1.1) with the structure (6.1)–(6.2). For example, we can prove a similar inequality as (4.1) for the case (6.1)–(6.2). Thus, it is easy to establish the contractive property as Corollary 4.2 for the case (6.1)–(6.2). Since the involved techniques for establishing the convergence are very similar as the case of VI (1.1) with (1.2)–(1.3), we omit the details of this extension.

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