

# On the Optimal Linear Convergence Rate of a Generalized Proximal Point Algorithm

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Received: 6 March 2016 / Revised: 18 May 2017 / Accepted: 1 June 2017 / Published online: 12 July 2017  
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**Abstract** The proximal point algorithm (PPA) has been well studied in the literature. In particular, its linear convergence rate has been studied by Rockafellar in 1976 under certain condition. We consider a generalized PPA in the generic setting of finding a zero point of a maximal monotone operator, and show that the condition proposed by Rockafellar can also sufficiently ensure the linear convergence rate for this generalized PPA. Indeed we show that these linear convergence rates are optimal. Both the exact and inexact versions of this generalized PPA are discussed. The motivation of considering this generalized PPA is that it includes as special cases the relaxed versions of some splitting methods that are originated from PPA. Thus, linear convergence results of this generalized PPA can be used to better understand the convergence of some widely used algorithms in the literature. We focus on the particular convex minimization context and specify Rockafellar's condition to see how to ensure the linear convergence rate for some efficient numerical schemes, including the classical augmented Lagrangian method proposed by Hensen and Powell in 1969 and its relaxed version, the original alternating direction method of multipliers (ADMM) by Glowinski and Marrocco in 1975 and its relaxed version (i.e., the generalized ADMM by Eckstein and Bertsekas in 1992). Some refined conditions weaker than existing ones are proposed in these particular contexts.

**Keywords** Convex programming · Proximal point algorithm · Augmented Lagrangian method · Alternating direction method of multipliers · Linear convergence rate

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Min Tao was supported by the Natural Science Foundation of China: NSFC-11301280, 11471156.

Xiaoming Yuan was supported by the General Research Fund from Hong Kong Research Grants Council: 12300515.

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### 1 Introduction

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ . A set-valued mapping  $T : H \rightarrow 2^H$  is said to be monotone if

$$\langle z - z', w - w' \rangle \geq 0, \quad \forall z, z' \in H, \quad w \in T(z), \quad w' \in T(z').$$

$T$  is said to be maximal monotone if, in addition, its graph is not properly contained in the graph of any other monotone operator. A fundamental problem is finding a zero point, denoted by  $z^*$ , of a maximal monotone set-valued mapping  $T$ :

$$0 \in T(z). \tag{1.1}$$

Throughout, the set of  $T$ 's zero point, denoted by  $zer(T)$ , is assumed to be nonempty.

The proximal point algorithm (PPA), which traces back to [25,27], has been playing an important role both theoretically and algorithmically for (1.1). Starting from an arbitrary point  $z^0$  in  $H$ , the PPA iteratively generates its sequence  $\{z^k\}$  by the scheme

$$0 \in c_k T(z^{k+1}) + z^{k+1} - z^k, \tag{1.2}$$

where  $\{c_k\}$ , called proximal parameter, is a sequence of positive real numbers. Indeed, as shown in [34], the convergence of PPA can be ensured when  $\{c_k\}$  is bounded away from zero. Moreover, an inexact version of PPA was proposed in [34], allowing the subproblem (1.2) to be solved approximately subject to some inexactness criteria. Conceptually, the inexact version of PPA can be written as

$$0 \approx c_k T(z^{k+1}) + z^{k+1} - z^k, \tag{1.3}$$

in which the accuracy should be judiciously chosen to guarantee its convergence. Let

$$J_{c_k T} := (I + c_k T)^{-1} \tag{1.4}$$

denote the resolvent operator of the maximal monotone set-valued mapping  $T$  for a positive scalar  $c_k$  (Note that  $J_{c_k T}$  is single-valued, see, e.g., [12]). Then, the exact and inexact versions of the PPA can be written, respectively, as

$$z^{k+1} = J_{c_k T}(z^k) \tag{1.5}$$

and

$$z^{k+1} \approx J_{c_k T}(z^k). \tag{1.6}$$

Technically, (1.6) includes (1.5) as the special case where the tolerance of accuracy is zero. But we still discuss them individually because (1.5) is of particular interest and it may have stronger convergence, because it requires estimating the resolvent operator accurately.

Research results on the convergence of PPA can be found in earlier literature. For example, when  $T$  is specified as the sum of a single-valued, monotone and hemicontinuous mapping and the normal cone to a bounded set, i.e., the problem (1.1) reduces to a variational inequality, then some convergence of the exact version of PPA (1.2) with  $c_k \equiv c$  in the weak topology was investigated in [25,26]. In [34], the convergence of both the exact and inexact versions of PPA was comprehensively studied; it is indeed the work [34] that popularized PPA in optimization community. More specifically, under the condition that  $\{c_k\}$  is bounded away from zero, the convergence of (1.6) [thus also (1.5)] in the weak topology was proved when the accuracy for “ $\approx$ ” in (1.6) is specified into certain forms (see (A) and (B) of Section 1 in [34]). In fact, the exact version (1.5) was shown to find a solution point of (1.1) after finitely many iterations in

[34]. Note that the strong monotonicity of  $T$  is not required for the analysis in [34]. Moreover, if the inverse of  $T$  (denoted by  $T^{-1}$ ) is Lipschitz continuous at 0 (see Definition 2.4 in Section 2 or Section 3 in [34]) and  $\{c_k\}$  is bounded away from zero, it was proved in [34] that the (1.6) [thus also (1.5)] with some relative error control in its accuracy is linearly convergent. There are many other articles studying the PPA from various perspectives. For example, the PPA application to nonconvex problems studied in [14], the PPA with variable metric in [6], a unified convergence rate analysis for some PPA-based decomposition methods in [36], accelerated PPA schemes with a worst-case  $O(1/k^2)$  convergence rate proposed in [19], the logarithmic quadratic proximal extension considered in [1, 2], and some other proximal-type algorithms in [38]. We particularly refer to [18, 28] for some insightful analysis on the iteration complexity of PPA, which can be regarded as a measure of its worst-case convergence rate. Algorithmically, the PPA is the basis of a large number of celebrated methods, e.g., the projected gradient method [31], the extragradient method [22], the extended extragradient and hyperplane schemes in [1], the forward-backward operator splitting method [24], and the accelerated projected gradient method [29].

As studied in [8, 12, 15, 17], the PPA schemes (1.5) and (1.6) can be generalized, respectively, as

$$\text{[Exact Version]} : \quad z^{k+1} = z^k - \gamma(z^k - J_{c_k T}(z^k)), \quad (1.7)$$

and

$$\text{[Inexact Version]} : \quad z^{k+1} \approx z^k - \gamma(z^k - J_{c_k T}(z^k)). \quad (1.8)$$

In (1.7) and (1.8), the proximal parameter sequence  $\{c_k\}$  is also required to be bound away from zero, i.e.,  $c_k \geq \kappa > 0$  for all  $k$ , and the relaxation factor  $\gamma \in (0, 2)$ . The generalized PPA schemes (1.7) and (1.8) usually can accelerate the original PPA schemes numerically, see, e.g., [3, 7, 13] for some numerical verifications. Thus, from the PPA perspective itself, it is interesting to consider its generalized versions. Another reason of considering the generalized PPA schemes (1.7) and (1.8) is that the original PPA scheme (1.5) indeed is a unified illustration of some different schemes for different models—it has been well studied that some popular iterative schemes such as the Douglas–Rachford splitting method (DRSM) in [11, 24], the Peaceman–Rachford splitting method in [24, 30] and the augmented Lagrangian method (ALM) in [21, 32] are all special cases of the PPA (1.5) with specific choices of  $T$ . Thus, generalizing the PPA scheme (1.5) (Resp., (1.6)) as (1.7) (Resp., (1.8)) represents a unified consideration for accelerating a series of well known splitting algorithms, especially in the convex optimization context. Let us just elaborate on the detail of the DRSM. Recall that (see [12], also Section 8 for details) the DRSM is a special case of the PPA (1.5). In [15], it was proved that the alternating direction method of multipliers (ADMM), which was originally proposed in [16] and now finds many applications in a wide range of areas, is a special case of the DRSM. Thus, the ADMM is also a special case of (1.5) and it can be accelerated immediately by the scheme (1.7). This application inspired the so-called generalized ADMM in [12], whose acceleration effectiveness was demonstrated recently in [13] by some statistical learning applications.

Our main purpose is to extend the analysis in [34] to the generalized PPA schemes (1.7) and (1.8), and establish their linear convergence rates under the same assumption as [34]:  $T^{-1}$  is Lipschitz continuous at 0. We further show that these linear convergence rates are indeed optimal. Because of the just-mentioned explanation, studying the linear convergence of the generalized schemes (1.7) and (1.8) helps us better understand the convergence properties of a number of specific splitting methods in the convex optimization context through a

unified perspective. In [15], the linear convergence of the exact version (1.7) with  $c_k \equiv c$  and  $\gamma \in (1, 2)$  was discussed under the assumptions that  $T$  is both strongly monotone (see Definition 2.3) and Lipschitz continuous. In [8], also under the assumption that  $T$  is strongly monotone, the sublinear and linear convergence rates of the schemes (1.7) and (1.8) with  $c_k \equiv c$  was studied; and these results were especially specified for the DRSM and PRSM scenarios. The results in [8] were then refined in [9] for the special DRSM and PRSM cases of (1.2). Note that, as analyzed in [34], “the assumption of Lipschitz continuity of  $T^{-1}$  at 0 turns out to be very natural in applications to convex programming”. Indeed, we will show later that this assumption is weaker than those considered in [8,9,15] (see the example in Sect. 2.2) and it suffices to ensure the linear convergence of the schemes (1.7) and (1.8) for the case  $\gamma \in (0, 2)$ . Thus, the distinction of this work from existing results in the literature is that stronger convergence rates are established under weaker conditions for the generalized PPA schemes (1.7) and (1.8). We will also consider several specific convex optimization contexts of the abstract model (1.1) and investigate how this assumption can be specified in these special contexts to ensure the linear convergence rates for some well-studied benchmark algorithms in the literature.

The rest of this paper is organized as follows. In Sect. 2, some preliminaries useful for further analysis are summarized. Then, we discuss the convergence and the linear convergence rate of the exact version of the generalized PPA (1.7) in Sect. 3. In Sect. 4, the convergence and linear convergence rate of its inexact version (1.8), in which the inexactness criterion is also specified, is studied. Then, we revisit the assumption “ $T^{-1}$  is Lipschitz continuous at 0” in Sect. 5 and show that it can be further relaxed. In Sect. 6, we discuss the possibility of deriving the superlinear convergence for the schemes (1.7) and (1.8). In Sect. 7, we apply the scheme (1.7) to a canonical convex minimization model with linear constraints and discuss the linear convergence for the resulting generalized ALM scheme. In Sect. 8, we focus on the analysis for the linear convergence of the ADMM and the generalized ADMM scheme, both are special cases of the scheme (1.7). Finally, some conclusions are made in Sect. 9.

## 2 Preliminaries

In this section, we recall some definitions and known results for further discussions.

### 2.1 Some Definitions

We first recall some basic definitions to be used in our analysis.

**Definition 2.1** Let  $T : H \rightarrow 2^H$  be set-valued and maximal monotone. Then,  $T$  is said to be nonexpansive if  $\|w' - w\| \leq \|z' - z\|, \forall z, z' \in H, w \in T(z), w' \in T(z')$ .

**Definition 2.2** Let  $T : H \rightarrow 2^H$  be set-valued and maximal monotone. Then,  $T$  is said to be firmly nonexpansive if  $\|w' - w\|^2 \leq \langle z' - z, w' - w \rangle, \forall z, z' \in H, w \in T(z), w' \in T(z')$ .

**Definition 2.3** Let  $T : H \rightarrow 2^H$  be set-valued and maximal monotone. Then,  $T$  is called  $\alpha$ -strongly monotone if  $\langle z - z', w - w' \rangle \geq \alpha \|z - z'\|^2, \forall z, z' \in H, w \in T(z), w' \in T(z')$  for  $\alpha > 0$ .

**Definition 2.4** Let  $T$  be set-valued and be defined on  $H$ . Then,  $T^{-1}$  is called Lipschitz continuous at  $\mathbf{0}$  with modulus  $a \geq 0$  if there is a unique solution  $z^*$  to  $\mathbf{0} \in T(z)$  (i.e.  $T^{-1}(\mathbf{0}) = \{z^*\}$ ), and for some  $\tau > 0$  we have  $\|z - z^*\| \leq a\|w\|$  whenever  $z \in T^{-1}(w)$  and  $\|w\| \leq \tau$ .

Definition 2.4 is quoted from [34]. Based on these definitions, we have some immediate conclusions. For examples, if  $T$  is nonexpansive, then it is Lipschitz continuous. Moreover, clearly, the problem (1.1) has a unique solution point when  $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$ .

### 2.2 An Example

Consider the problem (1.1), where  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by

$$T(x_1, x_2) := \frac{1}{a}(x_2, -x_1) \quad \text{with } a > 0. \tag{2.1}$$

Obviously, the operator  $T$  defined in (2.1) is maximal monotone and the solution point of (1.1) with (2.1) is  $z^* = (0, 0)$ . Moreover, it holds

$$\|T^{-1}(z_1) - T^{-1}(z_2)\| \leq a\|z_1 - z_2\|, \quad \forall z_1, z_2 \in \mathbb{R}^2, \tag{2.2}$$

and

$$\langle T(z_1) - T(z_2), z_1 - z_2 \rangle = 0, \quad \forall z_1, z_2 \in \mathbb{R}^2. \tag{2.3}$$

Thus,  $T^{-1}$  is Lipschitz continuous at  $0$  with modulus  $a > 0$  while  $T$  is not strongly monotone. This example indeed shows that the assumption “ $T^{-1}$  is Lipschitz continuity at  $\mathbf{0}$ ” is weaker than the strong monotonicity assumption on  $T$  as assumed in [8,9,15].

### 2.3 Some Known Results

Then, we summarize some known results that are relevant to our analysis. The following lemma summarizes some well-known properties of a firmly nonexpansive operator. The proof is straightforward and thus omitted, or see, e.g, [12].

**Lemma 2.5** *We have the following facts.*

- i) *All firmly nonexpansive operators are nonexpansive.*
- ii) *An operator  $T$  is firmly nonexpansive if and only if  $2T - I$  is nonexpansive.*
- iii) *An operator is firmly nonexpansive if and only if it is of the form  $\frac{1}{2}(C + I)$ , where  $C$  is nonexpansive.*
- iv) *An operator  $T$  is firmly nonexpansive if and only if  $I - T$  is firmly nonexpansive.*

In the following lemma, we show some simple conclusions for the resolvent operator of a maximal monotone operator.

**Lemma 2.6** *Let  $T : H \rightarrow 2^H$  be set-valued and maximal monotone;  $J_{cT}$  be defined in (1.4), and  $c > 0$  be a scalar. Then, we have*

- i)  $\langle J_{cT}(z) - J_{cT}(z'), (I - J_{cT})(z) - (I - J_{cT})(z') \rangle \geq 0, \forall z, z' \in H.$
- ii)  $\|z - z'\|^2 \geq \|J_{cT}(z) - J_{cT}(z')\|^2 + \|(I - J_{cT})(z) - (I - J_{cT})(z')\|^2, \forall z, z' \in H.$

*Proof* Obviously,  $J_{cT}$  defined in (1.4) is nonexpansive, and it implies the first property immediately. The second property is trivial because of Property (i). □

Last, let us recall the representation lemma, see, e.g., [12].

**Lemma 2.7** (The representation lemma) *Let  $c > 0$  and let  $T$  be monotone on  $H$ . Then every element  $z$  of  $H$  can be written in at most one way as  $x + cy$ , where  $y \in T(x)$ . If  $T$  is maximal, then every element  $z$  of  $H$  can be written in exactly one way as  $x + cy$ , where  $y \in T(x)$ .*

### 3 Convergence of the Exact Version (1.7)

In this section, we show that the generalized PPA (1.7) also converges linearly to a zero point of  $T$  under the assumption “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus”, the same one as that in [34]. For a lighter notation in analysis, we use the notation  $\tilde{z}^k = J_{c_k T}(z^k)$  in the following analysis.

#### 3.1 Global Convergence

First, we show the global convergence of (1.7). Note that we do not need the assumption “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus” for proving the global convergence. The next theorem shows that the sequence  $\{z^k\}$  generalized by (1.7) with  $\gamma \in (0, 2)$  is strictly contractive with respect to the solution set of (1.1), an important property ensuring its global convergence.

**Theorem 3.1** [12] (Strict contraction) *Let  $\{z^k\}$  be the sequence generated by the exact version of the generalized PPA scheme (1.7) with  $\gamma \in (0, 2)$  and  $\{c_k\}$  bounded away from 0; let  $z^*$  be a solution point of (1.1). We have*

$$\|z^{k+1} - z^*\|^2 \leq \|z^k - z^*\|^2 - \gamma(2 - \gamma) \left\| z^k - \tilde{z}^k \right\|^2. \tag{3.1}$$

Based on Theorem 3.1, the convergence of (1.7) can be easily established.

**Theorem 3.2** (Global convergence) *Let  $\{z^k\}$  be the sequence generated by the exact version of the generalized PPA scheme (1.7) with  $\gamma \in (0, 2)$  and  $\{c_k\}$  bounded away from 0. Then it converges in the weak topology to a solution point of (1.1).*

*Proof* According to (3.1), the sequence  $\{z^k\}$  is bounded, and it has at least one weak accumulation point, say  $z^\infty$ . Let  $\{z^{k_j}\}$  be the subsequence converging in the weak topology to  $z^\infty$ . Recall the notation  $\tilde{z}^k = J_{c_k T}(z^k)$  and the definition of  $J_{c_k T}$  in (1.4). We thus have

$$c_{k_j}^{-1}(z^{k_j} - \tilde{z}^{k_j}) \in T(\tilde{z}^{k_j}).$$

Then, using the monotonicity of  $T$ , for an integer  $k_j$ , it holds that

$$\langle z - \tilde{z}^{k_j}, w - c_{k_j}^{-1}(z^{k_j} - \tilde{z}^{k_j}) \rangle \geq 0, \quad \text{for all } z, w \text{ satisfying } w \in T(z). \tag{3.2}$$

Again, it follows from (3.1) that  $\lim_{j \rightarrow \infty} \|z^{k_j} - \tilde{z}^{k_j}\| = 0$ . Recall  $\{c_k\}$  is bounded away from 0. Then, taking  $j \rightarrow \infty$  in (3.2), we obtain

$$\langle z - z^\infty, w \rangle \geq 0 \text{ for all } z, w \text{ satisfying } w \in T(z).$$

In view of the maximality of  $T$ , this inequality implies that  $z^\infty$  is a solution point of (1.1), see, e.g. [34]. It is easy to see from Theorem 3.1 that the sequence  $\{z^k\}$  cannot have more than one accumulation point. Thus,  $\{z^k\}$  converges in the weak topology to  $z^\infty$  which a solution point of (1.1). The proof is complete. □

*Remark 1* If the real Hilbert space  $H$  is finite dimension, the result in this theorem can be improved to “strongly converge”.

### 3.2 Linear Convergence

Now, under the assumption “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus”, we prove the linear convergence of (1.7). First, two lemmas are presented.

**Lemma 3.3** *Let  $T : H \rightarrow 2^H$  be maximal monotone and  $z^*$  be a solution point of (1.1); let  $c_k > 0$ . If  $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with modulus  $a > 0$ , then there exists a positive  $\tau$  such that*

$$\|J_{c_k T}(z) - z^*\| \leq \frac{a}{\sqrt{a^2 + c_k^2}} \|z - z^*\| \text{ when } \|c_k^{-1}(z - J_{c_k T}(z))\| \leq \tau, \quad \forall z \in H. \quad (3.3)$$

*Proof* Applying Property (ii) in Lemma 2.6 with  $\tilde{z} = z^*$  and  $c = c_k$ , we get

$$\|z - z^*\|^2 \geq \|J_{c_k T}(z) - z^*\|^2 + \|(I - J_{c_k T})(z)\|^2. \quad (3.4)$$

Recall the definition of  $J_{c_k T}$  in (1.4). We have

$$c_k^{-1}(I - J_{c_k T})(z) \in T(J_{c_k T}(z)).$$

Since  $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with modulus  $a > 0$ , it follows from Definition 2.4 that there exists a positive parameter  $\tau$  such that

$$\|J_{c_k T}(z) - z^*\| \leq a \|c_k^{-1}(I - J_{c_k T})(z)\| \text{ when } \|c_k^{-1}(I - J_{c_k T})(z)\| \leq \tau.$$

Substituting this inequality into (3.4), we obtain (3.3). The proof is complete. □

*Remark 2* If some stronger assumptions such as “ $T$  is  $\frac{1}{a}$ -strongly monotone” hold as some existing work [8, 15], the assertion (3.3) can be easily improved as

$$\|J_{c_k T}(z) - z^*\| \leq \frac{a}{a + c_k} \|z - z^*\| \quad \forall z \in H. \quad (3.5)$$

Under the weaker assumption “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus”, however, the assertion (3.3) is optimal in the sense that the coefficient in the right-hand side cannot be smaller. To see this, let us consider the example (2.1) again in Sect. 2.2. It follows from (2.3) that

$$\langle c_k T(J_{c_k T}(z)) - c_k T(J_{c_k T}(z^*)), J_{c_k T}(z) - z^* \rangle = 0. \quad (3.6)$$

Consequently, we have

$$\begin{aligned} \|z - z^*\|^2 &= \|J_{c_k T}(z) - z^*\|^2 + \|z - J_{c_k T}(z)\|^2 + 2\langle z - J_{c_k T}(z), J_{c_k T}(z) - z^* \rangle \\ &= \|J_{c_k T}(z) - z^*\|^2 + \|c_k T(J_{c_k T}(z))\|^2 + 2\langle c_k T(J_{c_k T}(z)), J_{c_k T}(z) - z^* \rangle \\ &= \|J_{c_k T}(z) - z^*\|^2 + \|c_k T(J_{c_k T}(z))\|^2 \\ &\quad + 2\langle c_k T(J_{c_k T}(z)) - c_k T(J_{c_k T}(z^*)), J_{c_k T}(z) - z^* \rangle \\ &= \left(1 + \frac{c_k^2}{a^2}\right) \|J_{c_k T}(z) - z^*\|^2, \end{aligned} \quad (3.7)$$

in which the last inequality is because of the identity

$$\|c_k T(J_{c_k T}(z)) - c_k T(J_{c_k T}(z^*))\|^2 = \frac{c_k^2}{a^2} \|J_{c_k T}(z) - J_{c_k T}(z^*)\|^2$$

and the assertion (3.6). Therefore, the inequality (3.3) is tight and this indeed implies that the linear convergence rate to be established for (1.7) is optimal.

**Lemma 3.4** *Let  $\{z^k\}$  be the sequence generated by the exact version of the generalized PPA scheme (1.7) with  $\gamma \in (0, 2)$ , and  $z^*$  be a solution point of (1.1). If  $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with modulus  $a > 0$ , and the proximal parameter sequence  $\{c_k\}$  is bounded away from zero ( $c_k \geq \kappa > 0$  for any  $k$ ), then there exists an integer  $\hat{k}$  such that*

$$\|\tilde{z}^k - z^*\| \leq \frac{a}{\sqrt{a^2 + c_k^2}} \|z^k - z^*\| \quad \forall k > \hat{k}. \tag{3.8}$$

*Proof* It follows from Lemma 3.3, also can be derived from Theorem 2 of [34]. □

Now, we prove the local linear convergence rate of (1.7) in the following theorem.

**Theorem 3.5** *If  $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with modulus  $a > 0$  and the proximal parameter  $\{c_k\}$  is bounded away from zero ( $c_k \geq \kappa > 0$ ), then the sequence  $\{z^k\}$  generated by the exact version of the generalized PPA scheme (1.7) with  $\gamma \in (0, 2)$  converges strongly to  $z^*$ , the unique solution of (1.1). Moreover, there exists an index  $\hat{k}$  such that*

$$\|z^{k+1} - z^*\|^2 \leq \varrho \|z^k - z^*\|^2, \quad \forall k > \hat{k}, \tag{3.9}$$

with

$$\varrho := 1 - \min(\gamma, 2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2} \in (0, 1). \tag{3.10}$$

That is, the sequence  $\{z^k\}$  converges linearly to a solution point of (1.1).

*Proof* Simple algebra shows that

$$\begin{aligned} \|z^{k+1} - z^*\|^2 &= (1 - \gamma)^2 \|z^k - z^*\|^2 + \gamma^2 \|\tilde{z}^k - z^*\|^2 \\ &\quad + 2\gamma(1 - \gamma) \langle \tilde{z}^k - z^*, z^k - z^* \rangle \\ &= (1 - \gamma)^2 \|z^k - z^*\|^2 + (2\gamma - \gamma^2) \|\tilde{z}^k - z^*\|^2 \\ &\quad + 2\gamma(1 - \gamma) \langle \tilde{z}^k - z^*, z^k - \tilde{z}^k \rangle. \end{aligned} \tag{3.11}$$

Obviously, the assertion (3.9)–(3.10) follows directly from Lemma 3.4 when  $\gamma = 1$ . If  $0 < \gamma \leq 1$ , then it follows from Lemma 3.4 that

$$\begin{aligned} \|z^{k+1} - z^*\|^2 &\leq (1 - \gamma) \|z^k - z^*\|^2 + \gamma \|\tilde{z}^k - z^*\|^2 \\ &= \left( 1 - \gamma \frac{c_k^2}{a^2 + c_k^2} \right) \|z^k - z^*\|^2. \end{aligned}$$

On the other hand, applying the property (ii) in Lemma 2.6 with  $z = z^k$  and  $\tilde{z} = z^*$ , we get

$$\|z^k - z^*\|^2 \geq \|\tilde{z}^k - z^*\|^2 + \|z^k - \tilde{z}^k\|^2. \tag{3.12}$$

We thus have

$$\langle \tilde{z}^k - z^*, z^k - \tilde{z}^k \rangle \geq 0, \tag{3.13}$$

Moreover, if  $1 < \gamma < 2$ , because of (3.11), (3.13) and Lemma 3.4, we have

$$\|z^{k+1} - z^*\|^2 \leq \left( 1 - (2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2} \right) \|z^k - z^*\|^2.$$



To show (3.10), notice that  $\gamma \in (0, 2)$  and  $c_k \geq \kappa > 0$  for any  $k$ . Thus, we have

$$\begin{aligned} 0 < 1 - \min(\gamma, 2\gamma - \gamma^2) \leq \varrho &:= 1 - \min(\gamma, 2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2} \\ &< 1 - \min(\gamma, 2\gamma - \gamma^2) \frac{\kappa^2}{a^2 + \kappa^2} < 1. \end{aligned}$$

Therefore, the inequalities (3.9) and (3.10) imply the local linear convergence rate of the sequence  $\{z^k\}$ . The proof is complete.  $\square$

*Remark 3* For the problem (1.1) with  $T$  defined in (2.1), it follows from (3.11) and the analysis in Remark 2 that

$$\begin{aligned} \|z^{k+1} - z^*\|^2 &= (1 - \gamma)^2 \|z^k - z^*\|^2 + (2\gamma - \gamma^2) \|\tilde{z}^k - z^*\|^2 \\ &\quad + 2\gamma(1 - \gamma) \langle \tilde{z}^k - z^*, z^k - \tilde{z}^k \rangle \\ &= (1 - \gamma)^2 \|z^k - z^*\|^2 + (2\gamma - \gamma^2) \|\tilde{z}^k - z^*\|^2 \\ &= \left( 1 - (2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2} \right) \|z^k - z^*\|^2, \quad \forall \gamma \in (0, 2). \end{aligned} \tag{3.14}$$

Recall (3.9) and (3.10), this means the bound derived in Theorem 3.5 is tight at least for  $\gamma \in [1, 2)$ . Hence, the linear convergence established in Theorem 3.5 is optimal for the PPA scheme (1.7) with  $\gamma \in [1, 2)$ .

*Remark 4* For simplicity, we fix  $\gamma \in (0, 2)$  as a constant in our discussion. According to the proof of Theorem 3.5, it is trivial to see that the convergence result still holds when  $\gamma$  is permitted to vary iteratively and the sequence  $\{\gamma_k\}$  satisfies the condition  $\sum_k \gamma_k(2 - \gamma_k) = +\infty$ .

*Remark 5* Similarly as Definition 2.4, if a sequence  $\{z^k\}$  converges to  $z^*$ , we can define “ $T^{-1}$  is Lipschitz continuous with modulus  $a \geq 0$  at the sequence  $\{z^k\}$ ” if there exists some  $\tau > 0$  such that

$$\|z^k - z^*\| \leq a \|w^k\| \text{ whenever } z^k \in T^{-1}(w^k) \text{ and } \|w^k\| \leq \tau.$$

Then, it can be easily seen that the linear convergence of  $\{z^k\}$  generated by (1.7) can be guaranteed under the less strengthen assumption “ $T^{-1}$  is Lipschitz continuous at the iterates  $\{\tilde{z}^k\}$  with positive modulus when  $k$  is sufficiently large”. Recall the fact  $\tilde{z}^k \in T^{-1}(c_k^{-1}(z^k - \tilde{z}^k))$  and  $z^* \in T^{-1}(\mathbf{0})$ . Then, this less strengthen assumption is equivalent to saying that there exists an integer  $\hat{k}$  such that

$$\|\tilde{z}^k - z^*\| \leq a \|c_k^{-1}(z^k - \tilde{z}^k)\| \text{ when } k > \hat{k}, \tag{3.15}$$

where  $\hat{k}$  is large enough such that  $\|c_k^{-1}(z^k - \tilde{z}^k)\| \leq \tau$ . Note that  $\|c_k^{-1}(z^k - \tilde{z}^k)\| \leq \tau$  can be ensured by the fact  $\lim_{k \rightarrow \infty} \|z^k - \tilde{z}^k\| = 0$  implied in (3.1) and that  $\{c_k\}$  is bounded away from 0. More discussion is referred to Sect. 5.

### 4 Convergence of the Inexact Version (1.8)

In this section, we specify the inexactness criterion for (1.8) and show its linear convergence under the same assumption of “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus”.

This is a generalization of the inexact version (1.6) considered in [34]. More specifically, we consider the scheme

$$\begin{cases} z^{k+1} = (1 - \gamma)z^k + \gamma \tilde{z}^k, \\ \|\tilde{z}^k - J_{c_k T}(z^k)\| \leq \delta_k \|z^k - z^{k+1}\|, \end{cases} \tag{4.1}$$

where  $\gamma \in (0, 2)$ ,  $c_k \geq \kappa > 0$  for any  $k$ , and  $\{\delta_k\}$  is a sequence of positive real numbers satisfying  $\sum_k \delta_k < +\infty$ .

Note that in (4.1), we consider using relative errors to control the accuracy in (1.8); thus it is different from the inexact version in [8] which uses absolute errors. We still use the notation  $\tilde{z}^k = J_{c_k T}(z^k)$  in the upcoming analysis.

### 4.1 Global Convergence

Again, we first show the global convergence for the sequence  $\{z^k\}$  generated by (4.1). Note that we do not need the assumption “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus” for proving the global convergence. We first prove several lemmas for this purpose. Their proofs are elementary; but we still include them for completeness.

**Lemma 4.1** [41] *Let  $\{\alpha_k\}$  be a positive sequence satisfying  $\sum_{k=1}^{\infty} \alpha_k < +\infty$ . Then, we have*

$$\prod_{k=1}^{\infty} (1 + \alpha_k) < +\infty.$$

*Proof* Obviously, it holds that  $\log(1 + x) \leq x$  when  $0 < x < 1$ . Hence, we have

$$\sum_{k=1}^{\infty} \log(1 + \alpha_k) \leq \sum_{k=1}^{\infty} \alpha_k < +\infty,$$

which implies the assertion immediately. □

**Lemma 4.2** [41] *Let  $\{\delta_k\}$  be a positive sequence satisfying  $\sum_{k=1}^{\infty} \delta_k < +\infty$  and  $\gamma > 0$  be a constant. Then, we have*

$$\prod_{k=1}^{\infty} \frac{1 + \gamma \delta_k}{1 - \gamma \delta_k} < +\infty.$$

*Proof* Since  $\sum_{k=1}^{\infty} \delta_k < +\infty$ , we have  $\delta_k \rightarrow 0$  when  $k \rightarrow \infty$ . Thus, there exists an integer  $\hat{k}$  such that

$$1 - \gamma \delta_k \geq \frac{1}{2} \text{ when } k \geq \hat{k}. \tag{4.2}$$

Hence, we have

$$\begin{aligned} \prod_{k=1}^{\infty} \frac{1 + \gamma \delta_k}{1 - \gamma \delta_k} &= \prod_{k=1}^{\hat{k}-1} \frac{1 + \gamma \delta_k}{1 - \gamma \delta_k} \cdot \prod_{k=\hat{k}}^{\infty} \frac{1 + \gamma \delta_k}{1 - \gamma \delta_k} \\ &\leq \prod_{k=1}^{\hat{k}-1} \frac{1 + \gamma \delta_k}{1 - \gamma \delta_k} \cdot 2 \prod_{k=\hat{k}}^{\infty} (1 + \gamma \delta_k) < +\infty. \end{aligned}$$

The proof is complete. □

The proof of the following lemma can be found in, e.g., [39,40]; thus we omit it.

**Lemma 4.3** *Let  $\{a_k\}$  and  $\{b_k\}$  be positive sequences;  $\sum_{k=1}^\infty b_k < +\infty$ ; and*

$$a_{k+1} \leq a_k + b_k, \quad \forall k. \tag{4.3}$$

*Then, the sequence  $\{a_k\}$  is convergent.*

Now we start to prove the global convergence of (4.1). The key is the sequence generated by the inexact version (4.1) is asymptotically emerged with the sequence by the generalized PPA (1.7). With this fact, the convergence of (4.1) can be established easily.

**Theorem 4.4** *Let  $\{z^k\}$  be the sequence generated by the inexact version of the generalized PPA scheme (4.1). Then, we have*

- (1) *The sequence  $\{z^k\}$  is bounded.*
- (2) *It holds that*

$$\lim_{k \rightarrow \infty} \|z^k - \bar{z}^k\| = 0. \tag{4.4}$$

*Proof* Recall we use  $\bar{z}^k = J_{c_k T}(z^k)$  for easier notation. Let us use one more notation

$$\hat{z}^{k+1} := (1 - \gamma)z^k + \gamma\bar{z}^k.$$

Indeed,  $\hat{z}^{k+1}$  denotes the iterate generated by the exact version (1.7) from the given  $z^k$ . Thus, for an arbitrary solution point  $z^*$  of (1.1), it follows from (3.1) that

$$\|\hat{z}^{k+1} - z^*\|^2 \leq \|z^k - z^*\|^2 - \gamma(2 - \gamma)\|z^k - \bar{z}^k\|. \tag{4.5}$$

Recall the definition of  $z^{k+1}$  in (4.1). We have

$$z^{k+1} - z^{k+1} = \gamma(\bar{z}^k - \bar{z}^k), \tag{4.6}$$

where  $\bar{z}^k$  is also given in (4.1). Thus, for any solution point  $z^*$  of (1.1), we have

$$\begin{aligned} \|z^{k+1} - z^*\| &\leq \|z^{k+1} - \hat{z}^{k+1}\| + \|\hat{z}^{k+1} - z^*\| \\ &\leq \gamma\delta_k \|z^k - z^{k+1}\| + \|\hat{z}^{k+1} - z^*\| \\ &\leq \gamma\delta_k (\|z^k - z^*\| + \|z^{k+1} - z^*\|) + \|\hat{z}^{k+1} - z^*\| \\ &\leq \gamma\delta_k (\|z^k - z^*\| + \|z^{k+1} - z^*\|) + \|z^k - z^*\|, \end{aligned} \tag{4.7}$$

where the second inequality results from the inexact criterion in (4.1) and the last inequality follows from (4.5). Without loss of generality, we assume that  $1 - \gamma\delta_k > 0$  for all  $k$ . Then, we get

$$\|z^{k+1} - z^*\| \leq \frac{1 + \gamma\delta_k}{1 - \gamma\delta_k} \|z^k - z^*\| \leq \dots \leq \prod_{i=1}^k \frac{1 + \gamma\delta_i}{1 - \gamma\delta_i} \|z^0 - z^*\|.$$

Using Lemma 4.2, the sequence  $\{z^k\}$  is bounded. The first assertion is proved.

Now we prove the second assertion. Again, for an arbitrary solution point  $z^*$  of (1.1), since  $\{z^k\}$  is bounded and because of (4.5), there exists a positive scalar  $R$  such that

$$\|z^k - z^*\| < R, \quad \forall k \tag{4.8}$$

and

$$\|\hat{z}^k - z^*\| < R, \quad \forall k. \tag{4.9}$$

We thus have

$$\begin{aligned}
 \|z^{k+1} - z^*\|^2 &= \|\hat{z}^{k+1} - z^* + (z^{k+1} - \hat{z}^{k+1})\|^2 \\
 &= \|\hat{z}^{k+1} - z^*\|^2 + \|z^{k+1} - \hat{z}^{k+1}\|^2 + 2\langle \hat{z}^{k+1} - z^*, z^{k+1} - \hat{z}^{k+1} \rangle \\
 &\leq \|\hat{z}^{k+1} - z^*\|^2 + 2\|\hat{z}^{k+1} - z^*\| \|z^{k+1} - \hat{z}^{k+1}\| + \|z^{k+1} - \hat{z}^{k+1}\|^2 \\
 &\leq \|\hat{z}^{k+1} - z^*\|^2 + 2R\gamma\delta_k \|z^k - z^{k+1}\| + \gamma^2\delta_k^2 \|z^k - z^{k+1}\|^2 \\
 &\leq \|z^k - z^*\|^2 - \gamma(2 - \gamma)\|z^k - \bar{z}^k\|^2 + 4R^2\gamma\delta_k + 4R^2\gamma^2\delta_k^2, \tag{4.10}
 \end{aligned}$$

where the second inequality follows from (4.9) and (4.1); and the last inequality is because of (4.5) and (4.8). Moreover, since  $\gamma \in (0, 2)$ , we have

$$\|z^{k+1} - z^*\|^2 \leq \|z^k - z^*\|^2 + 4R^2\gamma\delta_k + 4R^2\gamma^2\delta_k^2,$$

and

$$\sum_k (4R^2\gamma\delta_k + 4R^2\gamma^2\delta_k^2) < +\infty.$$

Now, using Lemma 4.3 with  $a_k := \|z^k - z^*\|^2$  and  $b_k := 4R^2\gamma\delta_k + 4R^2\gamma^2\delta_k^2$ , we obtain

$$\lim_{k \rightarrow \infty} \|z^k - z^*\| =: A, \tag{4.11}$$

where  $A$  is a positive scalar. On the other hand, recall that  $\{\delta_k\}$  is summable, so is  $\{\delta_k^2\}$ . We thus have  $\sum_k \delta_k^2 < \infty$ . Then, it follows from (4.10) that

$$\gamma(2 - \gamma)\|z^k - \bar{z}^k\|^2 \leq \|z^k - z^*\|^2 - \|z^{k+1} - z^*\|^2 + 4R^2\gamma\delta_k + 4R^2\gamma^2\delta_k^2, \quad \forall k.$$

Then, we have  $\sum_k \|z^k - \bar{z}^k\|^2 < \infty$  and thus  $\lim_{k \rightarrow \infty} \|z^k - \bar{z}^k\| = 0$ . The proof is complete.  $\square$

Theorem 4.4 shows that the accuracy of iterates generated by the inexact version (4.1) is iteratively increased, which essentially implies the convergence of the sequence of (4.1). We provide the rigorous proof in the following theorem.

**Theorem 4.5** (Global convergence) *Let  $\{z^k\}$  be the sequence generated by the inexact version of the generalized PPA scheme (4.1). Then, it converges in the weak topology to a solution point of (1.1).*

*Proof* Since the sequence  $\{z^k\}$  is bounded, it has a weak accumulation point  $z^\infty$ . Let  $\{z_{k_j}\}$  be the subsequence converging in the weak topology to  $z^\infty$ . Note that

$$c_k^{-1}(z^k - J_{c_k T}(z^k)) \in T(J_{c_k T}(z^k)).$$

Thus, using the monotonicity of  $T$ , for any  $k$ , we have

$$\langle z - J_{c_k T}(z^k), w - c_k^{-1}(z^k - J_{c_k T}(z^k)) \rangle \geq 0, \text{ for all } z, w \text{ satisfying } w \in T(z).$$

Let  $k = k_j$  in the above inequality, take  $j \rightarrow \infty$ , and combine it with (4.4). We thus have

$$\langle z - z^\infty, w \rangle \geq 0, \text{ for all } z, w \text{ satisfying } w \in T(z).$$

which, together with the maximal monotonicity of  $T$ , means that  $z^\infty$  is a solution point of (1.1).

Finally, using (4.11), we can prove by contradiction that the sequence  $\{z^k\}$  has only one weak accumulation point (the proof is similar to that of Theorem 3 in [12]). Thus, the sequence

$\{z^k\}$  converges in the weak topology to  $z^\infty$  which is a solution point of (1.1). The proof is complete.  $\square$

*Remark 6* If the real Hilbert space  $H$  is finite dimension, the result in this theorem can be improved to “strongly converge”.

### 4.2 Linear Convergence

In this subsection, under the assumption “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus”, we prove the linear convergence for the sequence  $\{z^k\}$  generated by (4.1). Recall the notation  $\tilde{z}^k = J_{c_k T}(z^k)$ . The following lemma can be found in [34].

**Lemma 4.6** [34] *Let  $\{z^k\}$  be the sequence generated by the inexact version of the generalized PPA scheme (4.1) with  $\gamma \in (0, 2)$  and  $\sum_k \delta_k < +\infty$ . If  $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with modulus  $a > 0$ , then there exists an integer  $k_1$  such that*

$$\|\tilde{z}^k - z^*\| \leq \frac{a}{\sqrt{a^2 + c_k^2}} \|z^k - z^*\| \quad \forall k > k_1.$$

*Proof* From Theorem 4.4, we know that  $\lim_{k \rightarrow \infty} \|z^k - \tilde{z}^k\| = 0$ . Then, the conclusion follows immediately from the proof of Lemma 3.4.  $\square$

The main result of this subsection is summarized in the following theorem. This result reduces to Theorem 2 in [34] if  $\gamma = 1$ .

**Theorem 4.7** *Assume  $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with modulus  $a > 0$  and the proximal parameter sequence  $\{c_k\}$  is bounded away from zero ( $c_k \geq \kappa > 0$ ). Let  $\{z^k\}$  be the sequence generated by the inexact version of the generalized PPA scheme (4.1). Then, the sequence  $\{z^k\}$  converges strongly to the unique solution  $z^*$ . Moreover, there exists an integer  $\hat{k}$  such that*

$$\|z^{k+1} - z^*\| \leq \theta_k \|z^k - z^*\| \text{ when } k > \hat{k},$$

where  $z^*$  is a solution point of (1.1) and

$$0 < \theta_k := \frac{\sqrt{\left(1 - \min(\gamma, 2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2}\right)} + \gamma \delta_k}{1 - \gamma \delta_k} < 1, \text{ when } k > \hat{k}.$$

That is,  $\{z^k\}$  converges linearly to  $z^*$ .

*Proof* Recall in Theorem 4.5, it is proved that the sequence  $\{z^k\}$  converges to a solution point  $z^*$  of (1.1). First, it is easy to see that there exists an integer  $k_1$  such that

$$\|\hat{z}^{k+1} - z^*\|^2 \leq \left(1 - \min(\gamma, 2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2}\right) \|z^k - z^*\|^2, \quad k > k_1. \tag{4.12}$$

In addition, it follows from (4.7) that

$$\begin{aligned} \|z^{k+1} - z^*\| &\leq \gamma \delta_k (\|z^k - z^*\| + \|z^{k+1} - z^*\|) + \|\hat{z}^{k+1} - z^*\| \\ &\leq \gamma \delta_k (\|z^k - z^*\| + \|z^{k+1} - z^*\|) \\ &\quad + \sqrt{\left(1 - \min(\gamma, 2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2}\right)} \|z^k - z^*\|, \quad k > k_1, \end{aligned}$$

where the last inequality follows from (4.12). Accordingly, we have

$$\|z^{k+1} - z^*\| \leq \frac{\sqrt{1 - \min(\gamma, 2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2}} + \gamma \delta_k}{1 - \gamma \delta_k} \|z^k - z^*\| \text{ when } k > k_1.$$

Note that  $\delta_k \rightarrow 0$  and  $c_k \geq \kappa > 0$ . Then, there exists an integer  $\hat{k}$ , without loss of generality, assuming  $\hat{k} > k_1$ , such that

$$\theta_k := \frac{\sqrt{1 - \min(\gamma, 2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2}} + \gamma \delta_k}{1 - \gamma \delta_k} < 1, \text{ when } k > \hat{k}.$$

Hence,  $\{z^k\}$  converges linearly to  $z^*$ , a solution point of (1.1). The proof is complete.  $\square$

*Remark 7* Again, the results in Theorem 4.7 still hold if  $\gamma$  varies iteratively and the sequence  $\{\gamma_k\}$  satisfies the condition  $\sum_k \gamma_k(2 - \gamma_k) = +\infty$ .

*Remark 8* Similarly as Sect. 3.2, it can be seen from the proofs of Lemma 4.6 and Theorem 4.7 that the linear convergence of the sequence  $\{z^k\}$  generated by (4.1) can be guaranteed under the less strengthen condition “ $T^{-1}$  is Lipschitz continuous at the iterates  $\{z^k\}$  with positive modulus when  $k$  is large enough”.

### 5 Further Study on Assumption

Under the assumption “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus”, we have shown the linear convergence for both the exact version (1.7) and inexact version (4.1) of the generalized PPA. Recall that the generalized PPA (1.7) include the PPA (1.5) as a special case with  $\gamma = 1$  and our analysis extends the result in [34] for (1.5). In [8], the linear convergence of the generalized PPA (1.7) with  $c_k \equiv c$  has been studied under the assumption that  $T$  is  $\alpha$ -strongly monotone, which is stronger than “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus”.

In the following, we show that although we restrict our analysis under the assumption “ $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus”, theoretically this assumption can be further relaxed in order to ensure the linear convergence of (1.7) and (4.1). Note that the assertion in the following lemma does not depend on any specific iterative scheme.

**Theorem 5.1** *Let  $z^*$  be a solution point of (1.1) and the sequence  $\{c_k\}$  be both upper and below bounded, i.e.,  $0 < \kappa \leq c_k \leq \zeta$  for all  $k$ . If  $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$  with positive modulus  $a$ , then  $J_{c_k T}$  defined in (1.4) is Lipschitz continuous at  $z^*$  and  $\sup_k \{L_k\} < 1$ , where  $L_k$  is the Lipschitz constant of  $J_{c_k T}$ .*

*Proof* It follows from Lemma 3.3 that there exists  $\tau > 0$  such that

$$\|J_{c_k T}(z) - z^*\| \leq \frac{a}{\sqrt{a^2 + c_k^2}} \|z - z^*\|, \text{ when } \|c_k^{-1}(z - J_{c_k T}(z))\| \leq \tau. \tag{5.1}$$

Recall that  $z^* \in T^{-1}(\mathbf{0})$ ,  $J_{c_k T}(z) \in T^{-1}(c_k^{-1}(z - J_{c_k T}(z)))$  and  $T^{-1}$  is Lipschitz continuous at  $\mathbf{0}$ . We thus have

$$\|J_{c_k T}(z) - z^*\| \rightarrow 0, \text{ when } \|c_k^{-1}(z - J_{c_k T}(z))\| \rightarrow 0.$$

Since  $c_k \leq \zeta$  for all  $k$ , we have  $\frac{1}{\zeta} \|J_{\zeta T}(z) - z\| \leq \frac{1}{c_k} \|J_{c_k T}(z) - z\| \rightarrow 0$ . From above inequality, we see that  $\|z - z^*\| \rightarrow 0$  when  $c_k^{-1} \|z - J_{c_k T}(z)\| \rightarrow 0$ . Thus,  $J_{c_k T}$  is Lipschitz continuous at  $z^*$  with the constant  $L_k := \frac{a}{\sqrt{a^2 + c_k^2}} \leq \frac{a}{\sqrt{a^2 + \kappa^2}} < 1$  for any  $k$ , according to

(5.1). The proof is complete. □

**Theorem 5.2** *Suppose the sequence  $\{c_k\}$  is both upper and below bounded, that is, there exists constants  $\varsigma$  and  $\kappa$  such that  $0 < \kappa \leq c_k \leq \varsigma$  for all  $k$ . Let  $\{z^k\}$  be the sequence generated by the exact version of the generalized PPA (1.7) or the inexact version (4.1). If  $J_{c_k T}$  is Lipschitz continuous at  $z^*$  with the constant  $L_k$ , and  $L_G := \sup_k \{L_k\} < 1$ , then*

- (1)  $T^{-1}$  is Lipschitz continuous at all the iterates  $\{\tilde{z}^k\}$  with positive modulus when  $k$  is sufficiently large.
- (2)  $\{z^k\}$  converges linearly to a solution point of (1.1).

*Proof* For a solution point of (1.1),  $z^*$ , we have  $z^* = J_{c_k T}(z^*)$ . Recall the notation  $\tilde{z}^k = J_{c_k T}(z^k)$ . Thus, it holds that

$$\begin{aligned} \|z^k - \tilde{z}^k\| &= \|(z^k - z^*) - (J_{c_k T}(z^k) - J_{c_k T}(z^*))\| \\ &\geq \|z^k - z^*\| - \|J_{c_k T}(z^k) - J_{c_k T}(z^*)\| \\ &\geq (1 - L_G) \|z^k - z^*\|, \end{aligned} \tag{5.2}$$

which implies

$$\frac{1}{(1 - L_G)^2} \|z^k - \tilde{z}^k\|^2 \geq \|z^k - z^*\|^2 \geq \|\tilde{z}^k - z^*\|^2 + \|z^k - \tilde{z}^k\|^2. \tag{5.3}$$

Then, it follows from the above inequality and  $0 < c_k \leq \varsigma$  that

$$\begin{aligned} \|\tilde{z}^k - z^*\|^2 &\leq \frac{2L_G - L_G^2}{(1 - L_G)^2} \|z^k - \tilde{z}^k\|^2 \\ &\leq \frac{2L_G - L_G^2}{(1 - L_G)^2} \varsigma^2 \|c_k^{-1}(z^k - \tilde{z}^k)\|^2. \end{aligned} \tag{5.4}$$

According to Theorems 3.1 and 4.4, for the sequence  $\{z^k\}$  generated by either the exact version (1.7) or the inexact version (4.1), we have  $\lim_{k \rightarrow \infty} \|z^k - z^*\| = 0$ . Since  $c_k \geq \kappa > 0$ , there exists an integer  $\hat{k}$  such that

$$\|c_k^{-1}(z^k - \tilde{z}^k)\| \leq \kappa^{-1} \|z^k - \tilde{z}^k\| \leq \tau \text{ when } k > \hat{k},$$

where  $\tau > 0$  is a given constant. Note the facts  $\tilde{z}^k \in T^{-1}(c_k^{-1}(z^k - \tilde{z}^k))$  and  $z^* \in T^{-1}(0)$ . Consequently, it follows from (5.4) that  $T^{-1}$  is Lipschitz continuous at all the iterates  $\{\tilde{z}^k\}$

with modulus  $a := \frac{\varsigma \sqrt{2L_G - L_G^2}}{1 - L_G}$  when  $k$  is large enough.

Now, we prove (2). Indeed, as commented in Remarks 5 and 8, the linear convergence of the schemes (1.7) and (4.1) can be ensured since  $T^{-1}$  is Lipschitz continuous at all the iterates  $\{\tilde{z}^k\}$  with positive modulus when  $k$  is sufficiently large and  $\{c_k\}$  is below bounded. Thus, the assertion (2) is proved. The proof is complete. □

So far, we have mentioned various conditions including strongly convexity in [8], the assumption in [34] and the one in Theorems 5.1 and 5.2, to ensure the linear convergence of

the schemes (1.7) and (4.1). In Fig. 1, we show their relationships for the special case where  $c_k \equiv c$  for all  $k$ , which is clearly an interesting choice for implementing the schemes (1.7) and (4.1).

Finally, we indicate that the linear convergence results both in Theorems 3.5 and 4.7 still hold when the condition that  $T^{-1}$  is Lipschitz continuous at 0” is replaced with the assumption “ $T$  is metrically subregular at  $z^* \in T^{-1}(\mathbf{0})$ ” [23,37]. Let us take Lemma 3.3, which is a key lemma in our analysis, as an example to illustrate how to extend our results to the case with the relaxed assumption.

**Lemma 5.3** *Let  $T : H \rightarrow 2^H$  be maximal monotone and  $z^*$  be a solution point of  $\mathbf{0} \in T(z)$ ; let  $c_k > 0$ . If  $T$  is metrically subregular at  $z^* \in T^{-1}(\mathbf{0})$  with modulus  $a > 0$ , then there exists a neighborhood  $U$  of  $z^*$  such that*

$$\text{dist}(J_{c_k T}(z), T^{-1}(\mathbf{0})) \leq \frac{a}{\sqrt{a^2 + c_k^2}} \text{dist}(z, T^{-1}(\mathbf{0})) \text{ when } z \in U. \tag{5.5}$$

*Proof* Applying Property (ii) in Lemma 2.6 with  $\tilde{z} = z^*$  (any  $z^* \in T^{-1}(\mathbf{0})$ ) and  $c = c_k$ , we get

$$\|z - z^*\|^2 \geq \|J_{c_k T}(z) - z^*\|^2 + \|(I - J_{c_k T})(z)\|^2. \tag{5.6}$$

We have

$$c_k^{-1}(I - J_{c_k T})(z) \in T(J_{c_k T}(z)).$$

Hence,

$$\|z - J_{c_k T}(z)\| \geq c_k \min\{\|z\| \mid z \in T(J_{c_k T}(z))\} = c_k \text{dist}(\mathbf{0}, T(J_{c_k T}(z))). \tag{5.7}$$

Since  $T$  is metrically subregular at  $z^* \in T^{-1}(\mathbf{0})$  with modulus  $a > 0$ , it follows that there exists a neighborhood  $U$  of  $z^*$  such that

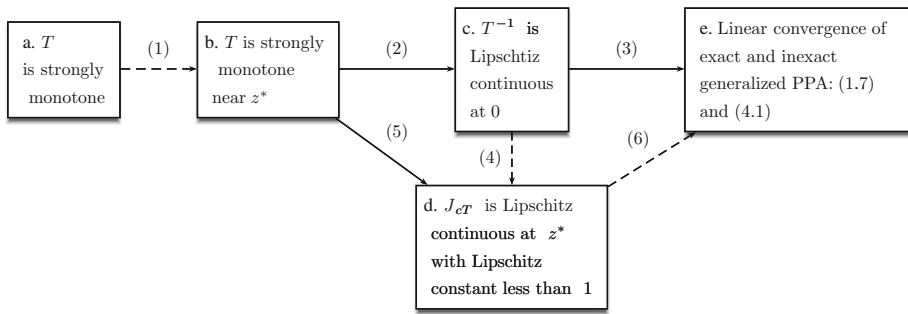
$$\begin{aligned} & \text{dist}(J_{c_k T}(z), T^{-1}(\mathbf{0}))^2 \\ & \leq \|J_{c_k T}(z) - P_{T^{-1}(\mathbf{0})}(z)\|^2 \text{ where } P_{T^{-1}(\mathbf{0})}(\cdot) \text{ represents the projection} \\ & \stackrel{(5.6)}{\leq} \|z - P_{T^{-1}(\mathbf{0})}(z)\|^2 - \|z - J_{c_k T}(z)\|^2 \\ & \stackrel{(5.7)}{\leq} \text{dist}(z, T^{-1}(\mathbf{0}))^2 - c_k^2 \text{dist}(\mathbf{0}, T(J_{c_k T}(z)))^2 \\ & \stackrel{*}{\leq} \text{dist}(z, T^{-1}(\mathbf{0}))^2 - \frac{c_k^2}{a^2} \text{dist}(J_{c_k T}(z), T^{-1}(\mathbf{0}))^2 \text{ when } z \in U. \end{aligned} \tag{5.8}$$

The last inequality (\*) is because of the metric subregularity of  $T$  at  $z^* \in T^{-1}(\mathbf{0})$  and  $J_{c_k T}(z) \in U$  whenever  $z \in U$ . With rearrangements, the proof is complete.  $\square$

## 6 Discussion on the Superlinear Convergence

In [34], under the assumption that “ $T^{-1}$  is Lipschitz continuous at 0 with positive modulus”, it was shown that the special case of (4.1) with  $\gamma = 1$  is superlinearly convergent if the proximal parameter  $c_k \rightarrow \infty$ . See Theorem 2 in [34]. One may ask if we can extend the same superlinear convergence result to (4.1) with a general  $\gamma$  in  $(0, 2)$ . In this section, we





**Fig. 1** Relationships among different conditions for linear convergence of (1.7) and (4.1)

take a closer look at this issue and give a negative answer to this question. It is sufficient to just analyze the exact version (1.7) to answer this question.

Recall (3.9) and (3.10). We have

$$\|z^{k+1} - z^*\|^2 \leq \left(1 - \min(\gamma, 2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2}\right) \|z^k - z^*\|^2. \tag{6.1}$$

As mentioned, this inequality is tight when the maximal monotone operator  $T$  is defined as (2.1). Note that  $\min(\gamma, 2\gamma - \gamma^2) \frac{c_k^2}{a^2 + c_k^2} \rightarrow 1$  when  $c_k \rightarrow \infty$  and  $\gamma = 1$ . Moreover, we have

$$0 < \min(\gamma, 2\gamma - \gamma^2) < 1 \text{ and } 0 < \frac{c_k^2}{a^2 + c_k^2} \leq 1$$

when  $\gamma \in (0, 2)$  and  $\gamma \neq 1$ . Thus, if  $c_k \rightarrow \infty$ , the coefficient in (6.1) goes to 0 only when  $\gamma = 1$ . This excludes the hope of establishing the superlinear convergence for the exact version of the generalized PPA (1.7) with  $\gamma \neq 1$  even when  $c_k \rightarrow \infty$ .

### 7 Application to ALM

Previously, we have discussed the linear convergence rates for the generalized PPA schemes (1.7) and (1.8) in the generic setting of (1.1) where  $T$  is an abstract maximal operator. In this and next sections, we specify our discussion to some special convex minimization models and discuss the linear convergence rates for two important algorithms which can be obtained by specifying the exact version of the generalized PPA scheme (1.7). For succinctness, discussions for their inexact counterparts stemming from the inexact version (1.8) are omitted. We refer to, e.g., [23], for some convergence rate analysis in terms of the iteration complexity for several splitting algorithms. The analysis therein is based on the inexact Krasosel’skii-Mann iteration from the nonexpansive operator perspective.

Let us first recall some known results and summarize them in the following two lemmas. The proof of the first lemma can be found in [33], and the proof of the second is trivial.

**Lemma 7.1** *Let  $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$  be closed, proper and convex. Then, we have*

- i) *If  $f$  is  $\mu_f$ -strongly convex, then  $f^*$  is differentiable and  $\nabla f^*$  is  $(1/\mu_f)$ -Lipschitz continuous.*

ii) If  $f$  is differentiable and  $\nabla f$  is  $L_f$ -Lipschitz continuous, then  $f^*$  is  $1/L_f$ -strongly convex.

**Lemma 7.2** *Let  $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$  be closed, proper and strongly convex; let  $\partial f$  be the subdifferential of  $f$ . Then,  $\partial f$  is strongly monotone.*

### 7.1 Preliminaries of ALM

First, we consider a canonical convex minimization model with linear constraints:

$$\begin{aligned} \min f(x) \\ \text{s.t. } Ax = b. \end{aligned} \tag{7.1}$$

where  $f : R^n \rightarrow (-\infty, \infty]$  is a closed and convex function,  $A \in R^{m \times n}$  and  $b \in R^m$ .

For solving (7.1), a benchmark is the augmented Lagrangian method (ALM) originally proposed in [21, 32]. Its iterative scheme reads as

$$\begin{cases} x^{k+1} = \arg \min_x \{f(x) - \langle p^k, Ax \rangle + \frac{1}{2}c_k \|Ax - b\|^2\}, \\ p^{k+1} = p^k - c_k(Ax^{k+1} - b), \end{cases} \tag{7.2}$$

where  $p^k$  is the Lagrange multiplier and  $c_k > 0$  is the penalty parameter of the linear constraints. As analyzed in [34], the dual problem of (7.1) is

$$\max_p \{-f^*(A^\top p) + \langle b, p \rangle\}, \tag{7.3}$$

where “\*” denotes the conjugate of a function, see [34]. Thus, solving (7.3) is equivalent to

$$0 \in S_A(p) := A \cdot \partial f^* \cdot (A^\top p) - b, \tag{7.4}$$

which is a specific application of the generic setting of (1.1) with  $T = S_A$ . In [35], it was precisely analyzed that the ALM scheme (7.2) is an application of the PPA (1.2) to the dual problem (7.4). Also in [34], it was mentioned that the mapping  $S_A(p)$  defined in (7.4) is maximal monotone.

### 7.2 A Generalized ALM

Following the analysis in [34], it is easy to see that if we apply the generalized PPA scheme (1.7) to (7.4), we can obtain a generalized ALM scheme as follows

$$\begin{cases} x^{k+1} = \arg \min_x \{f(x) - \langle p^k, Ax \rangle + \frac{1}{2}c_k \|Ax - b\|^2\}, \\ p^{k+1} = p^k - \gamma c_k(Ax^{k+1} - b), \end{cases} \tag{7.5}$$

which differs from the original ALM (7.2) in that there is a parameter  $\gamma \in (0, 2)$  for updating the Lagrange multiplier  $p^{k+1}$ . The details are presented in the following theorem.

**Theorem 7.3** *The generalized ALM scheme (7.5) is an application of the exact version of the generalized PPA (1.7) to (7.4).*

*Proof* The proof can be referred to that of Lemma 8.1 with the special consideration of  $g = 0$  and  $B = 0$ . Thus, we omit it. □

### 7.3 Linear Convergence of ALM schemes

Below, we show some conditions that can sufficiently ensure that the mapping  $S_{\mathcal{A}}^{-1}$  ( $S_{\mathcal{A}}$  defined in (7.4)) is Lipschitz continuous at 0 with positive modulus, and thus guarantee the linear convergence rate of the generalized ALM (7.5) (also the original ALM (7.2) if taking  $\gamma = 1$  in (7.5)).

**Theorem 7.4** *Let  $S_{\mathcal{A}}$  be defined in (7.4) and  $\{p^k\}$  be the sequence generated by the generalized ALM scheme (7.5). For the model (7.1), if  $f$  is convex and twice continuously differentiable at a neighborhood of  $x^*$ , and the matrix  $A$  is full row rank. Then, we have*

- (1) *The mapping  $S_{\mathcal{A}}$  is strongly monotone at a neighborhood of  $p^*$  ( $p^*$  is a zero point of  $S_{\mathcal{A}}(p)$  defined in (7.4)).*
- (2) *The mapping  $S_{\mathcal{A}}^{-1}$  exists and it is Lipschitz continuous at the origin with a positive modulus.*
- (3) *The sequence  $\{p^k\}$  converges locally on a linear rate to a zero point of  $S_{\mathcal{A}}$ .*

*Proof* (1) Since  $f$  is twice continuously differentiable at a neighborhood of  $x^*$ ,  $\nabla f$  is  $L_f$ -Lipschitz continuous locally. It follows from Property (ii) of Lemma 7.1 that  $f^*$  is  $\frac{1}{L_f}$ -strongly convex locally. Then, according to Lemma 7.2, we know that  $\partial f^*$  is  $\frac{1}{L_f}$ -strongly monotone locally. For any  $p, p' \in N(p^*, \delta) \subseteq R^m; w \in S_{\mathcal{A}}(p)$  and  $w' \in S_{\mathcal{A}}(p')$ ; there exist  $u \in \partial f^* \cdot (A^\top p)$  and  $u' \in \partial f^* \cdot (A^\top p')$  such that  $w = Au - b$  and  $w' = Au' - b$ . We thus have

$$\begin{aligned} \langle w - w', p - p' \rangle &= \langle A(u - u'), p - p' \rangle = \langle u - u', A^\top(p - p') \rangle \\ &\geq \frac{1}{L_f} \|A^\top(p - p')\|^2 \geq \frac{1}{L_f} \lambda_{\min}(AA^\top) \|p - p'\|^2, \end{aligned}$$

in which  $\lambda_{\min}(AA^\top)$  is the minimal eigenvalue of  $AA^\top$  and it holds that  $\lambda_{\min}(AA^\top) > 0$  because  $A$  is assumed to be full row rank. Note that the first inequality above is because of the  $\frac{1}{L_f}$ -strong monotonicity of  $\partial f^*$  at the neighborhood  $N(A^\top p^*, \delta') \subseteq R^n$ . Thus, it follows from Definition 2.3 that the mapping  $S_{\mathcal{A}}$  is strongly monotone locally. For assertion (2), it is an immediate conclusion based on Definitions 2.3 and 2.4. For assertion (3), it is a conclusion of Theorem 3.5 with  $T = S_{\mathcal{A}}$ . The proof is complete.  $\square$

*Remark 9* Note that the full-row-rank assumption of the matrix  $A$  in Theorem 7.4 can be replaced with other assumptions such as “the Hessian matrix  $\nabla^2 f(x^*)$  is positive definite in the null space of  $A^\top$ ”, i.e., the strong second-order condition as analyzed in [35] and many others, provided that more assumptions such as the strict complementary condition are further assumed for the model (7.1) per se. This discussion is beyond the scope of this paper, and we omit the analysis in details.

### 8 Application to ADMM

In this section, we consider another convex minimization model with a separable objective function:

$$\min_x \{f(x) + g(Mx)\} \tag{8.1}$$

where  $f : R^n \rightarrow (-\infty, \infty]$  and  $g : R^m \rightarrow (-\infty, \infty]$  are closed and convex functions, and the matrix  $M \in R^{m \times n}$ . Again, we only focus on the specification of the exact version of the

generalized PPA (1.7) and discuss how to ensure its linear convergence rate for the particular convex minimization context (8.1).

### 8.1 Preliminaries of ADMM

One particular case of (8.1) with a wide range of applications is where the functions  $f$  and  $g$  have their own properties and it is necessary to treat them individually in algorithmic design. For this purpose, we can reformulate (8.1) as

$$\begin{aligned} \min \quad & f(x) + g(w) \\ \text{s.t.} \quad & Mx = w, \end{aligned} \tag{8.2}$$

where  $w \in R^m$  is an auxiliary variable. For solving (8.2), a benchmark is the ADMM scheme originally proposed in [16]. The ADMM scheme for (8.2) reads as

$$\begin{cases} x^{k+1} = \arg \min_x \{f(x) + \langle p^k, Mx \rangle + \frac{1}{2}\lambda \|Mx - w^k\|^2\}, \\ w^{k+1} = \arg \min_w \{g(w) - \langle p^k, w \rangle + \frac{1}{2}\lambda \|Mx^{k+1} - w\|^2\}, \\ p^{k+1} = p^k + \lambda(Mx^{k+1} - w^{k+1}), \end{cases} \tag{8.3}$$

where  $p^k$  is the Lagrange multiplier and  $\lambda > 0$  is a penalty parameter of the linear constraints in (8.2).

Next, we recall some results in [12, 15] to demonstrate that the ADMM is indeed a special case of the PPA (1.5). All the details can be found in [12]. First, the dual of (8.2) is

$$\max_{p \in R^m} -\{f^*(-M^\top p) + g^*(p)\} \tag{8.4}$$

where “ $f^*$ ” and “ $g^*$ ” denote the conjugate of the convex functions  $f$  and  $g$ , respectively. Let

$$\mathcal{A} := \partial[f^* \cdot (-M^\top)] \text{ and } \mathcal{B} := \partial g^*. \tag{8.5}$$

As shown in [33], both  $\mathcal{A}$  and  $\mathcal{B}$  defined in (8.5) are maximal monotone. Then, (8.4) can be written as

$$0 \in \mathcal{A}(p) + \mathcal{B}(p). \tag{8.6}$$

We use  $J_{\lambda \mathcal{A}}$  and  $J_{\lambda \mathcal{B}}$  to denote the resolvent operators of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Moreover, we denote

$$G_{\lambda, \mathcal{A}, \mathcal{B}} = J_{\lambda \mathcal{A}}(2J_{\lambda \mathcal{B}} - I) + (I - J_{\lambda \mathcal{B}}) \tag{8.7}$$

and

$$S_{\lambda, \mathcal{A}, \mathcal{B}} := G_{\lambda, \mathcal{A}, \mathcal{B}}^{-1} - I. \tag{8.8}$$

As shown in [12],  $S_{\lambda, \mathcal{A}, \mathcal{B}}$  is maximal monotone when  $\mathcal{A}$  and  $\mathcal{B}$  are both maximal monotone. Indeed, the definition of  $S_{\lambda, \mathcal{A}, \mathcal{B}}$  can be expressed as

$$S_{\lambda, \mathcal{A}, \mathcal{B}} = \{(v + \lambda b, u - v) | (u, b) \in \mathcal{B}, (v, a) \in \mathcal{A}, v + \lambda a = u - \lambda b\}, \tag{8.9}$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are defined in (8.5). Moreover, let  $p^*$  be an solution point of (8.6) and  $z^*$  a solution point of

$$0 \in S_{\lambda, \mathcal{A}, \mathcal{B}}(z), \tag{8.10}$$

and let the sequence  $\{z^k\}$  be iteratively represented by

$$z^{k+1} = J_{\lambda, \mathcal{A}}((2J_{\lambda, \mathcal{B}} - I)(z^k)) + (I - J_{\lambda, \mathcal{B}})(z^k). \tag{8.11}$$

Indeed, (8.11) is exactly the application of the Douglas–Rachford splitting method (DRSM) in [11, 24] to (8.6). According to [12], we know some conclusions such as: (1) If  $z^*$  is a solution point of (8.10), then we have  $p^* := J_{\lambda, \mathcal{B}}(z^*)$  is a solution point of (8.6); and (2) If  $p^*$  is a solution point of (8.6) and  $(x^*, w^*)$  is a solution point of (8.2), then we have  $x^* \in \partial f^* \cdot (-M^\top p^*)$  and  $w^* \in \partial g^*(p^*)$ .

Applying the scheme (1.7) with  $T = S_{\lambda, \mathcal{A}, \mathcal{B}}$ , we obtain the exact version of the generalized PPA scheme

$$z^{k+1} = z^k - \gamma(z^k - J_{S_{\lambda, \mathcal{A}, \mathcal{B}}}(z^k)) \text{ with } \gamma \in (0, 2). \tag{8.12}$$

Indeed, via (8.12), the following exact version of the generalized ADMM scheme proposed in [12] can be recovered

$$\begin{cases} x^{k+1} = \arg \min_x \{f(x) + \langle p^k, Mx \rangle + \frac{1}{2}\lambda \|Mx - w^k\|^2\}, \\ w^{k+1} = \arg \min_w \{g(w) - \langle p^k, w \rangle + \frac{1}{2}\lambda \|\gamma Mx^{k+1} + (1 - \gamma)w^k - w\|^2\}, \\ p^{k+1} = p^k + \lambda(\gamma Mx^{k+1} + (1 - \gamma)w^k - w^{k+1}). \end{cases} \tag{8.13}$$

In the following, we elucidate the relationship between the sequence  $\{(x^k, w^k, p^k)\}$  generated by the generalized ADMM (8.13) and  $\{z^k\}$  represented by (8.12); and demonstrate that the generalized ADMM (8.13) can be written compactly as (8.12). The following lemma also clearly shows that the generalized ADMM (8.13) is an application of the generalized PPA (1.7) with  $T = S_{\lambda, \mathcal{A}, \mathcal{B}}$  and  $c_k \equiv 1$  to (8.10).

**Lemma 8.1** *Let  $\{(x^k, w^k, p^k)\}$  be generated by the generalized ADMM (8.13) and  $\{z^k\}$  be represented by (8.12); the operator  $S_{\lambda, \mathcal{A}, \mathcal{B}}$  be defined in (8.8). Assume that the initial points satisfy with  $z^0 = p^0 + \lambda w^0$  and  $p^0 = J_{\lambda, \mathcal{B}}(z^0)$ . Then, it holds that  $z^k = p^k + \lambda w^k$  and  $p^k = J_{\lambda, \mathcal{B}}(z^k)$  for all iterates.*

*Proof* The proof can be found in [12] (Theorem 8 therein). □

Finally, let us first present a lemma; its proof can be found in [24].

**Lemma 8.2** *The operator  $G_{\lambda, \mathcal{A}, \mathcal{B}}$  defined in (8.7) is firmly nonexpansive and it satisfies*

$$\begin{aligned} \langle G_{\lambda, \mathcal{A}, \mathcal{B}}(z) - G_{\lambda, \mathcal{A}, \mathcal{B}}(z'), z - z' \rangle &\geq \|G_{\lambda, \mathcal{A}, \mathcal{B}}(z) - G_{\lambda, \mathcal{A}, \mathcal{B}}(z')\|^2 \\ &\quad + \langle (I - J_{\lambda, \mathcal{B}})(z) - (I - J_{\lambda, \mathcal{B}})(z'), J_{\lambda, \mathcal{B}}(z) - J_{\lambda, \mathcal{B}}(z') \rangle, \\ &\forall z, z' \in H. \end{aligned} \tag{8.14}$$

### 8.2 When Does the Assumption Hold?

Based on our previous analysis, it is clear that the linear convergence of the generalized ADMM (8.13) can be ensured by the assumption “The mapping  $S_{\lambda, \mathcal{A}, \mathcal{B}}^{-1}(S_{\lambda, \mathcal{A}, \mathcal{B}}$  defined in (8.8) exists and it is Lipschitz continuous at 0 with positive modulus”. When the specific model (8.1) is considered, it is interesting to discern sufficient conditions that can ensure this assumption and thus guarantee the linear convergence of the generalized ADMM scheme (8.13); this is the main purpose of this subsection. We also refer to, e.g., [5, 10, 20] for discussions on the linear convergence of the original ADMM (8.3) for some special cases.

In the following, we show one scenario that can sufficiently ensure the mentioned assumption for the specific model (8.1) and thus guarantee the linear convergence of the the sequence  $\{z^k\}$  represented by (8.12).

**Theorem 8.3** For the model (8.1), if the function  $g$  is differentiable and strongly convex, and  $\nabla g$  is Lipschitz continuous near a solution point, then we have

- (1) The operator  $\mathcal{B} := \nabla g^*$  is both strongly monotone and Lipschitz continuous near the solution point.
- (2) The mapping  $S_{\lambda, \mathcal{A}, \mathcal{B}}^{-1}$  is Lipschitz continuous at the iterate  $\tilde{z}^k := J_{S_{\lambda, \mathcal{A}, \mathcal{B}}}(z^k)$  with positive modulus when  $k$  stays large enough.
- (3) The sequence  $\{z^k\}$  represented by (8.12) converges linearly to a solution point of (8.10).

*Proof* (1) According to Lemma 7.1, we know that  $g^*$  is differentiable. Thus,  $\mathcal{B} := \nabla g^*$  is both strongly monotone and Lipschitz continuous near the solution point of (8.4) according to Lemmas 7.1 and 7.2. Thus, the first conclusion is proved.

- (2) Next, we show that the Lipschitz constant of the operator  $G_{\lambda, \mathcal{A}, \mathcal{B}}$  is less than 1. Note that  $\frac{1}{\lambda}((I - J_{\lambda \mathcal{B}})(z)) \in \mathcal{B}(J_{\lambda \mathcal{B}}(z))$ . Let us assume that the strongly monotone modulus of  $\mathcal{B}$  is  $\alpha$ . That is,

$$\begin{aligned} &\langle (I - J_{\lambda \mathcal{B}})(z) - (I - J_{\lambda \mathcal{B}})(z'), J_{\lambda \mathcal{B}}(z) - J_{\lambda \mathcal{B}}(z') \rangle \\ &\geq \lambda \alpha \|J_{\lambda \mathcal{B}}(z) - J_{\lambda \mathcal{B}}(z')\|^2, \forall z, z' \in H. \end{aligned}$$

Moreover, let us assume that Lipschitz continuous constant of  $\mathcal{B}$  is  $\beta$ . Then, we have

$$\begin{aligned} \|z - z'\|^2 &= \|J_{\lambda \mathcal{B}}(z) - J_{\lambda \mathcal{B}}(z') + \lambda \mathcal{B}(J_{\lambda \mathcal{B}}(z)) - \lambda \mathcal{B}(J_{\lambda \mathcal{B}}(z'))\|^2 \\ &\leq (1 + \lambda \beta)^2 \|J_{\lambda \mathcal{B}}(z) - J_{\lambda \mathcal{B}}(z')\|^2, \forall z, z' \in H. \end{aligned}$$

Combining these two inequalities, we get

$$\begin{aligned} &\langle (I - J_{\lambda \mathcal{B}})(z) - (I - J_{\lambda \mathcal{B}})(z'), J_{\lambda \mathcal{B}}(z) - J_{\lambda \mathcal{B}}(z') \rangle \\ &\geq \frac{\lambda \alpha}{(1 + \lambda \beta)^2} \|z - z'\|^2, \forall z, z' \in H. \end{aligned} \tag{8.15}$$

Then, it follows from Lemma 8.2 and (8.15) that

$$\begin{aligned} \|z - z'\|^2 &\geq \langle G_{\lambda, \mathcal{A}, \mathcal{B}}(z) - G_{\lambda, \mathcal{A}, \mathcal{B}}(z'), z - z' \rangle \\ &\geq \|G_{\lambda, \mathcal{A}, \mathcal{B}}(z) - G_{\lambda, \mathcal{A}, \mathcal{B}}(z')\|^2 + \langle (I - J_{\lambda \mathcal{B}})(z) \\ &\quad - (I - J_{\lambda \mathcal{B}})(z'), J_{\lambda \mathcal{B}}(z) - J_{\lambda \mathcal{B}}(z') \rangle \\ &\geq \|G_{\lambda, \mathcal{A}, \mathcal{B}}(z) - G_{\lambda, \mathcal{A}, \mathcal{B}}(z')\|^2 + \frac{\lambda \alpha}{(1 + \lambda \beta)^2} \|z - z'\|^2, \forall z, z' \in H, \end{aligned}$$

where the first inequality follows from the non-expansiveness of the operator  $G_{\lambda, \mathcal{A}, \mathcal{B}}$ ; the second inequality is because of (8.14) and the last inequality holds because of (8.15). Consequently, we prove that

$$\|G_{\lambda, \mathcal{A}, \mathcal{B}}(z) - G_{\lambda, \mathcal{A}, \mathcal{B}}(z')\| \leq \sqrt{1 - \frac{\lambda \alpha}{(1 + \lambda \beta)^2}} \|z - z'\|, \forall z, z' \in H.$$

Recall the definitions of the strongly monotonicity and the Lipschitz continuity of  $\mathcal{B}$ . We have  $\alpha \leq \beta$  and thus the above inequality means the fact that the Lipschitz continuity constant of the operator  $G_{\lambda, \mathcal{A}, \mathcal{B}}$  is less than 1. Finally, it follows from Theorem 5.2 with  $T = S_{\lambda, \mathcal{A}, \mathcal{B}}$ ,  $c_k \equiv 1$  and  $G = G_{\lambda, \mathcal{A}, \mathcal{B}}$  that the mapping  $S_{\lambda, \mathcal{A}, \mathcal{B}}^{-1}$  is Lipschitz continuous at the iterate  $\{\tilde{z}^k\}$  with positive modulus when  $k$  stays large enough, where  $\tilde{z}^k := J_{S_{\lambda, \mathcal{A}, \mathcal{B}}}(z^k)$ .

- (3) Finally, the linear convergence of the sequence  $\{z^k\}$  follows assertion (2) of Theorem 5.2 with  $c_k \equiv 1$  and assertion (2) immediately. The proof is complete.  $\square$

Note that the linear convergence of  $\{z^k\}$  represented by (8.12) can be easily specified as the linear convergence of the generalized ADMM scheme (8.13) in terms of the variables in (8.2) and its dual. We summarize the specifications in the following corollary and omit the proofs.

**Corollary 8.4** *When the sequence  $\{z^k\}$  represented by (8.12) converges linearly to a solution point of (8.10), we have*

- (1) *The sequence  $\{p^k\}$  converges  $R$ -linearly to a solution point  $p^*$  of the dual problem (8.4).*
- (2) *The sequence  $\{w^k\}$  converges  $R$ -linearly to a solution point  $w^*$  of the primal problem (8.2).*
- (3) *The sequence  $\{Mx^k\}$  converges  $R$ -linearly to  $Mx^*$ , where  $x^*$  is a solution point of the primal problem (8.1). Moreover, if  $M$  is full column rank, then the sequence  $\{x^k\}$  converges  $R$ -linearly to  $x^*$ .*

*Remark 10* Under one of the following conditions, we can also prove the conclusion “The mapping  $S_{\lambda, \mathcal{A}, \mathcal{B}}^{-1}$  ( $S_{\lambda, \mathcal{A}, \mathcal{B}}$  defined in (8.8)) exists and it is Lipschitz continuous at 0 with positive modulus”. We omit the proof because it is analogous to that of Theorem 8.3.

- (1) The matrix  $M$  is full row rank, the function  $f$  is strongly convex and  $\nabla f$  is Lipschitz continuous near  $x^*$ , where  $x^*$  is a solution point of (8.1).
- (2) The matrix  $M$  is full row rank, the function  $f$  is convex and  $g$  is strongly convex near  $Mx^*$ , and  $\nabla f$  is Lipschitz continuous near  $x^*$ , where  $x^*$  is a solution point of (8.1).
- (3) The function  $f$  is strongly convex near  $x^*$  and  $g$  is convex, and  $\nabla g$  is Lipschitz continuous near  $Mx^*$ , where  $x^*$  is a solution point of (8.1).

Together with the condition in Theorem 8.3, these conditions coincide with the conditions in [9] (when  $B = -I$  and  $b = 0$  in the model (2) therein) to ensure the linear convergence of the generalized ADMM (8.13) for solving (8.1). In other words, the assumption “The mapping  $S_{\lambda, \mathcal{A}, \mathcal{B}}^{-1}$  ( $S_{\lambda, \mathcal{A}, \mathcal{B}}$  defined in (8.8)) exists and it is Lipschitz continuous at 0 with positive modulus” is weaker than these conditions.

*Remark 11* In [8], the linear convergence of the generalized ADMM (8.13) for solving (8.1) is ensured under the following assumptions: (1)  $M$  is full rank,  $f$  is convex and differentiable,  $\nabla f$  is Lipschitz continuous, and  $g$  is strongly convex; (2)  $f$  is strongly convex,  $g$  is convex and differentiable, and  $\nabla g$  is Lipschitz continuous. We here give some less strengthen conditions.

## 9 Conclusion

In this paper, we extend the condition in [34] that can ensure the linear convergence of the proximal point algorithm (PPA) to a generalized PPA scheme. Both the exact and inexact versions of the generalized PPA are studied, and their linear convergence rates are established under the same condition as the original PPA in [34]. We specifically consider two convex optimization models and study the linear convergence rates for generalized versions of the benchmark augmented Lagrangian method (ALM) and the alternating direction method of multipliers (ADMM), both are special cases of the proposed generalized PPA. Some concrete conditions are specified in the convex optimization contexts. It is interesting to find that the condition in [34] turns out to be still weaker than most of the existing conditions in the literature that were proposed to ensure the linear convergence for various specific forms of the PPA. This study provides a unified understanding of the linear convergence of a family

of operator splitting methods which have found a board spectrum of applications in various areas. These methods include the mentioned ALM, ADMM, their generalized and inexact versions, the Douglas–Rachford splitting method, the Peaceman–Rachford splitting method, and their generalized versions. Last, it is worthwhile to mention that our discussion can be easily extended to the case with weaker assumptions such as the metric subregularity considered in, e.g., [37].

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